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Global attractivity and positive almost periodic solution of a multispecies discrete mutualism system with time delays

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Abstract— **In this paper, we consider an almost periodic multispecies discrete Lotka-Volterra mutualism system with time delays. We first obtain the permanence and global attractivity of the system. By means of an almost periodic functional hull theory and constructing a suitable Lyapunov function, sufficient conditions are obtained for the existence of a unique strictly positive almost periodic solution which is globally attractive. An example together with numerical simulation indicates the feasibility of the main results.**

Index Terms—**Almost periodic solution, Mutualism system, Discrete, Global attractivity**

I. Introduction

Recently, investigating the almost periodic solutions of discrete and continuous population dynamics model with time delays has more extensively practical application value(see [1–19] and the references cited therein). In this paper, we are concerned with the following multispecies discrete Lotka-Volterra mutualism system with time delays

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k) - b_i(k)x_i(k-\sigma_i) + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{x_j(k-\tau_{ij})}{d_{ij}(k)+x_j(k-\tau_{ij})}\right\},\quad i=1,2,\cdots,n, \tag{1.1}$$

where $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_{ij}(k)\}$ and $\{d_{ij}(k)\}$ are bounded nonnegative almost periodic sequences such that

$$0 < a_i^l \le a_i(k) \le a_i^u, \qquad 0 < b_i^l \le b_i(k) \le b_i^u,$$

$$0 < c_{ij}^l \le c_{ij}(k) \le c_{ij}^u, \qquad 0 < d_{ij}^l \le d_{ij}(k) \le d_{ij}^u,$$

$i, j = 1, 2, \cdots, n, j \neq i,\ k \in \mathbf{Z}$. For any bounded sequence $\{f(k)\}$ defined on $\mathbf{Z}$, $f^u = \sup_{k\in\mathbf{Z}} f(k), f^l = \inf_{k\in\mathbf{Z}} f(k)$.

By the biological meaning, we will focus our discussion on the positive solutions of system (1.1). So it is assumed that the initial conditions of system (1.1) are the form:

$$x_i(\theta) = \varphi_i(\theta) \ge 0,\quad \varphi_i(0) > 0,$$
$$\theta \in N[-\tau, 0] = \{-\tau, -\tau+1, \ldots, 0\},$$
$$\tau = \max_{1\le i,j\le n, j\neq i}\{\sigma_i, \tau_{ij}\}. \tag{1.2}$$

To the best of our knowledge, this is the first paper to investigate the global stability of positive almost periodic solution of multispecies discrete Lotka-Volterra mutualism system with time delays. The aim of this paper is to obtain sufficient conditions for the existence of a unique globally attractive almost periodic solution of the systems (1.1) with initial condition (1.2), by utilizing an almost periodic functional hull theory and constructing a suitable Lyapunov functional and applying the analysis technique of papers [3, 12, 13].

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In the next section, we establish the permanence of system (1.1). Sufficient conditions for the global attractivity of system (1.1) are showed in Section 4. Then, in Section 5, we establish sufficient conditions to ensure the existence of a unique strictly positive almost periodic solution, which is globally attractive. The main result is illustrated by an example with a numerical simulation in the last section.

II. Preliminaries

First, we give the definitions of the terminologies involved.

Definition 2.1([20]) A sequence x: Z→R is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : |\, x(n+\tau) - x(n) \,| < \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all ε > 0; that is, for any given ε > 0, there exists an integer l(ε) > 0 such that each interval of length l(ε) contains an integer τ∈E{ε, x} with

$$|\, x(n+\tau) - x(n) \,| < \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of x(n).

Definition 2.2([21]) Let D be an open subset of Rm, f : Z ×D →R^m. f(n, x) is said to be almost periodic in n uniformly for x ∈ D if for any ε > 0 and any compact set S ⊂ D, there exists a positive integer l=l(ε, S) such that any interval of length l contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| < \varepsilon, \quad \forall (n, x) \in Z \times S.$$

τ is called an ε-translation number of f(n,x).

Definition 2.3([22]) The hull of f, denoted by H(f), is defined by

$$H(f) = \{g(n,x) : \lim_{k\to\infty} f(n+\tau_k, x) = g(n,x) \text{ uniformly on } Z\times S\},$$

for some sequence $\{\tau_k\}$, where S is any compact set in D.

Definition 2.4 Suppose that X(k) = (x_1(k), x_2(k), · · · , x_n(k)) is any solution of system (1.1). X(k) is said to be a strictly positive solution in Z if for k∈Z and i = 1, 2, · · · , n

$$0 < \inf_{k\in Z} x_i(k) \le \sup_{k\in Z} x_i(k) < \infty.$$

Now, we state several lemmas which will be useful in proving our main result.

Lemma 2.1([23]) {x(n)} is an almost periodic sequence if and only if for any integer sequence { k_i' }, there exists a subsequence $\{k_i\} \subset \{k_i^{'}\}$ such that the sequence $\{x(n+k_i)\}$ converges uniformly for all $n \in Z$ as $i \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.2([24]) Assume that sequence {x(n)} satisfies x(n) > 0 and

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\}$$

for $n \in N$, where a(n) and b(n) are non-negative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n) \le \frac{1}{b^l}\exp\{a^u - 1\}.$$

Lemma 2.3([24]) Assume that sequence {x(n)} satisfies

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \ge N_0,$$

$$\limsup_{n\to+\infty} x(n) \le x^*$$

and $x(N_0)>0$, where a(n) and b(n) are non-negative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{n\to+\infty} x(n) \ge \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}.$$

III. PERMANENCE

In this section, we establish the permanence result for system (1.1).

Theorem 3.1 System (1.1) with initial condition (1.2) is permanent, that is, there exist positive constants mi and M_i(i = 1, 2, · · · , n) which are independent of the solutions of system (1.1), such that for any positive solution ($x_1(k)$, $x_2(k)$, · · · , $x_n(k)$) of system (1.1), one has:

$$m_i \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_i, \qquad i = 1, 2, \cdots, n.$$

Proof. Let ($x_1(k)$, $x_2(k)$, · · · , $x_n(k)$) be any positive solution of system (1.1) with initial condition (1.2). From the first equation of system (1.1) it follows that

$$\begin{aligned} x_i(k+1) &\le x_i(k)\exp\Big\{a_i(k) + \sum_{j=1,j\ne i}^{n} c_{ij}(k)\Big\} \\ &\le x_i(k)\exp\Big\{a_i^u + \sum_{j=1,j\ne i}^{n} c_{ij}^u\Big\}. \end{aligned} \tag{3.1}$$

By using (3.1), one could easily obtain that

$$x_i(k-\sigma_i) \ge x_i(k)\exp\Big\{-\sigma_i\big(a_i^u + \sum_{j=1,j\ne i}^{n} c_{ij}^u\big)\Big\}. \tag{3.2}$$

Substituting (3.2) into the first equation of system (1.1), it follows that

$$x_i(k+1) \le x_i(k)\exp\Big\{a_i^u + \sum_{j=1,j\ne i}^{n} c_{ij}^u - b_i^l\exp\big\{-\sigma_i\big(a_i^u + \sum_{j=1,j\ne i}^{n} c_{ij}^u\big)\big\}x_i(k)\Big\}. \tag{3.3}$$

Thus, as a direct corollary of Lemma 2.2, according to (3.3), one has

$$\limsup_{k\to+\infty} x_i(k) \le \frac{1}{b_i^l}\exp\{(a_i^u + \sum_{j=1,j\ne i}^{n} c_{ij}^u)(\sigma_i+1) - 1\} \triangleq M_i. \tag{3.4}$$

For any small positive constant ε > 0, from (3.4) it follows that there exists a positive constants K > 0 such that for all k > K and i = 1, 2, · · · , n,

$$x_i(k) \le M_i + \varepsilon. \tag{3.5}$$

For $k \ge K+\sigma_i$, from (3.5) and system (1.1), we have

$$\begin{aligned} x_i(k+1) &\ge x_i(k)\exp\{a_i(k) - b_i(k)x_i(k-\sigma_i)\} \\ &\ge x_i(k)\exp\{a_i^l - b_i^u(M_i+\varepsilon)\}. \end{aligned} \tag{3.6}$$

Thus, by using (3.6) we obtain

$$x_i(k-\sigma_i) \le x_i(k)\exp\{-\sigma_i[a_i^l - b_i^u(M_i+\varepsilon)]\}. \tag{3.7}$$

Substituting (3.7) into system (1.1), for $k \ge K+\sigma_i$, it follows that

$$x_i(k+1) \ge x_i(k)\exp\Big\{a_i^l - b_i^u\exp\{-\sigma_i[a_i^l - b_i^u(M_i+\varepsilon)]\}x_i(k)\Big\}. \tag{3.8}$$

Thus, as a direct corollary of Lemma 2.3, according to (3.4) and (3.8), one has

$$\liminf_{k\to+\infty} x_i(k) \ge \min\{m_{i_1\varepsilon}, m_{i_2\varepsilon}\}, \tag{3.9}$$

where

$$\begin{aligned} m_{i_1\varepsilon} &= \frac{a_i^l}{b_i^u}\exp\{\sigma_i[a_i^l - b_i^u(M_i+\varepsilon)]\}, \\ m_{i_2\varepsilon} &= m_{i_1\varepsilon}\exp\Big\{a_i^l - b_i^u\exp\{-\sigma_i[a_i^l - b_i^u(M_i+\varepsilon)]\}M_i\Big\}. \end{aligned}$$

Letting ε→0, it follows that

$$\liminf_{k\to+\infty} x_i(k) \ge \frac{1}{2}\min\{m_{i_1}, m_{i_2}\} \triangleq m_i > 0, \tag{3.10}$$

where

$$\begin{aligned} m_{i_1} &= \frac{a_i^l}{b_i^u}\exp\{\sigma_i(a_i^l - b_i^u M_i)\}, \\ m_{i_2} &= m_{i_1}\exp\Big\{a_i^l - b_i^u\exp\{-\sigma_i(a_i^l - b_i^u M_i)\}M_i\Big\}. \end{aligned}$$

Then, (3.4) and (3.10) show that system (1.1) is permanent. The proof is completed.

IV. GLOBAL ATTRACTIVITY

In this section, by constructing a non-negative Lyapunov-like functional, we will obtain sufficient conditions for global attractivity of positive solutions of system (1.1) with initial condition (1.2). We first introduce a definition and prove a theorem which will be useful to obtain our main result.

Definition 4.1 A solution ($x_1(k)$, $x_2(k)$, · · · , $x_n(k)$) of system (1.1) with initial condition (1.2) is said to be globally attractive if for any other solution ($x^*_1(k)$, $x^*_2(k)$, · · · , $x^*_n(k)$) of system (1.1) with initial condition (1.2), we have

$$\lim_{k\to+\infty}(x_i^*(k) - x_i(k)) = 0, \quad i = 1, 2, \cdots, n.$$

Lemma 4.1 For any two positive solutions ($x_1(k)$, $x_2(k)$, · · · , $x_n(k)$) and ($x^*_1(k)$, $x^*_2(k)$, · · · , $x^*_n(k)$) of system (1.1) with initial condition (1.2), we have for $k \ge 2\tau$

$$\begin{aligned} \ln\frac{x_i(k+1)}{x_i^*(k+1)} &= \ln\frac{x_i(k)}{x_i^*(k)} - b_i(k)[x_i(k) - x_i^*(k)] \\ &+ \sum_{j=1,j\ne i}^{n} c_{ij}(k)d_{ij}(k)\frac{x_j(k-\tau_{ij}) - x_j^*(k-\tau_{ij})}{[d_{ij}(k) + x_j(k-\tau_{ij})][d_{ij}(k) + x_j^*(k-\tau_{ij})]} \\ &+ b_i(k)\sum_{s=k-\sigma_i}^{k-1}\Big\{[x_i(s) - x_i^*(s)]A_i(s)\big[a_i(s) - b_i(s)x_i^*(s-\sigma_i) + \sum_{j=1,j\ne i}^{n} c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s) + x_j^*(s-\tau_{ij})}\big] \\ &+ x_i(s)B_i(s)\Big[\sum_{j=1,j\ne i}^{n} c_{ij}(s)d_{ij}(s)\frac{x_j(s-\tau_{ij}) - x_j^*(s-\tau_{ij})}{[d_{ij}(s) + x_j(s-\tau_{ij})][d_{ij}(s) + x_j^*(s-\tau_{ij})]} \\ &- b_i(s)[x_i(s-\sigma_i) - x_i^*(s-\sigma_i)]\Big]\Big\}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} A_i(s) &= \exp\Big\{\theta_i(s)\big[a_i(s) - b_i(s)x_i^*(s-\sigma_i) + \sum_{j=1,j\ne i}^{n} c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s) + x_j^*(s-\tau_{ij})}\big]\Big\}, \\ B_i(s) &= \exp\Big\{\varphi_i(s)\big[a_i(s) - b_i(s)x_i(s-\sigma_i) + \sum_{j=1,j\ne i}^{n} c_{ij}(s)\frac{x_j(s-\tau_{ij})}{d_{ij}(s) + x_j(s-\tau_{ij})}\big] \\ &\quad + (1-\varphi_i(s))\big[a_i(s) - b_i(s)x_i^*(s-\sigma_i) + \sum_{j=1,j\ne i}^{n} c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s) + x_j^*(s-\tau_{ij})}\big]\Big\}, \end{aligned} \tag{4.2}$$

$\theta_i(s), \varphi_i(s) \in (0,1),\ i = 1, 2, \cdots, n.$

Proof. For i = 1, 2, · · · , n, we can have from system (1.1)

$$\ln\frac{x_i(k+1)}{x_i^*(k+1)}-\ln\frac{x_i(k)}{x_i^*(k)}=\ln\frac{x_i(k+1)}{x_i(k)}-\ln\frac{x_i^*(k+1)}{x_i^*(k)}$$
$$=a_i(k)-b_i(k)x_i(k-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\frac{x_j(k-\tau_{ij})}{d_{ij}(k)+x_j(k-\tau_{ij})}$$
$$-\Big[a_i(k)-b_i(k)x_i^*(k-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(k)\frac{x_j^*(k-\tau_{ij})}{d_{ij}(k)+x_j^*(k-\tau_{ij})}\Big]$$
$$=\sum_{j=1,j\neq i}^{n}c_{ij}(k)\Big[\frac{x_j(k-\tau_{ij})}{d_{ij}(k)+x_j(k-\tau_{ij})}-\frac{x_j^*(k-\tau_{ij})}{d_{ij}(k)+x_j^*(k-\tau_{ij})}\Big]-b_i(k)[x_i(k-\sigma_i)-x_i^*(k-\sigma_i)]$$
$$=\sum_{j=1,j\neq i}^{n}c_{ij}(k)d_{ij}(k)\frac{x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})}{[d_{ij}(k)+x_j(k-\tau_{ij})][d_{ij}(k)+x_j^*(k-\tau_{ij})]}-b_i(k)[x_i(k)-x_i^*(k)]$$
$$+b_i(k)\{[x_i(k)-x_i(k-\sigma_i)]-[x_i^*(k)-x_i^*(k-\sigma_i)]\},$$

that is

$$\ln\frac{x_i(k+1)}{x_i^*(k+1)}=\ln\frac{x_i(k)}{x_i^*(k)}+\sum_{j=1,j\neq i}^{n}c_{ij}(k)d_{ij}(k)\frac{x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})}{[d_{ij}(k)+x_j(k-\tau_{ij})][d_{ij}(k)+x_j^*(k-\tau_{ij})]}$$
$$-b_i(k)[x_i(k)-x_i^*(k)]+b_i(k)\{[x_i(k)-x_i(k-\sigma_i)]-[x_i^*(k)-x_i^*(k-\sigma_i)]\}. \tag{4.3}$$

Since

$$[x_i(k)-x_i(k-\sigma_i)]-[x_i^*(k)-x_i^*(k-\sigma_i)]$$
$$=\sum_{s=k-\sigma_i}^{k-1}[x_i(s+1)-x_i(s)]-\sum_{s=k-\sigma_i}^{k-1}[x_i^*(s+1)-x_i^*(s)]$$
$$=\sum_{s=k-\sigma_i}^{k-1}\{[x_i(s+1)-x_i^*(s+1)]-[x_i(s)-x_i^*(s)]\}, \tag{4.4}$$

and for k≥2τ

$$[x_i(s+1)-x_i^*(s+1)]-[x_i(s)-x_i^*(s)]$$
$$=x_i(s)\exp\big[a_i(s)-b_i(s)x_i(s-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(s)\frac{x_j(s-\tau_{ij})}{d_{ij}(s)+x_j(s-\tau_{ij})}\big]$$
$$-x_i^*(s)\exp\big[a_i(s)-b_i(s)x_i^*(s-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s)+x_j^*(s-\tau_{ij})}\big]-[x_i(s)-x_i^*(s)]$$
$$=[x_i(s)-x_i^*(s)]\Big\{\exp\big[a_i(s)|-b_i(s)x_i^*(s-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s)+x_j^*(s-\tau_{ij})}\big]-1\Big\}$$
$$+x_i(s)\Big\{\exp\big[a_i(s)-b_i(s)x_i(s-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(s)\frac{x_j(s-\tau_{ij})}{d_{ij}(s)+x_j(s-\tau_{ij})}\big]$$
$$-\exp\big[a_i(s)-b_i(s)x_i^*(s-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s)+x_j^*(s-\tau_{ij})}\big]\Big\}.$$

Using the Mean Value Theorem, we get

$$[x_i(s+1)-x_i^*(s+1)]-[x_i(s)-x_i^*(s)]$$
$$=[x_i(s)-x_i^*(s)]A_i(s)\Big[a_i(s)-b_i(s)x_i^*(s-\sigma_i)+\sum_{j=1,j\neq i}^{n}c_{ij}(s)\frac{x_j^*(s-\tau_{ij})}{d_{ij}(s)+x_j^*(s-\tau_{ij})}\Big]$$
$$+x_i(s)B_i(s)\Big[\sum_{j=1,j\neq i}^{n}c_{ij}(s)d_{ij}(s)\frac{x_j(s-\tau_{ij})-x_j^*(s-\tau_{ij})}{[d_{ij}(s)+x_j(s-\tau_{ij})][d_{ij}(s)+x_j^*(s-\tau_{ij})]}$$
$$-b_i(s)[x_i(s-\sigma_i)-x_i^*(s-\sigma_i)]\Big], \tag{4.5}$$

here $A_i(s)$, $B_i(s)$ are defined by (4.2). Then from (4.3)-(4.5), we can easily obtain (4.1). The proof is completed.

Theorem 4.1 Assume that in system (1.1) with initial condition (1.2), there exist positive constants β_i(i =1, 2, · · · , n) and $\eta > 0$ such that

$$\beta_iE_i-\sum_{j=1,j\neq i}^{n}\beta_jF_{ij}\geq\eta,\quad i=1,2,\cdots,n, \tag{4.6}$$

where

$$E_i=\min\{b_i^l,\frac{2}{M_i}-b_i^u\}-\sigma_iM_i(b_i^u)^2B_i^u-\sigma_ib_i^uA_i^u\Big(a_i^u+b_i^uM_i+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}^u}{d_{ij}^l}\Big),$$
$$F_{ij}=\frac{c_{ij}^u}{d_{ij}^l}(1+\sigma_jM_jb_j^uB_j^u). \tag{4.7}$$

Then for any two positive solutions ($x_1(k)$, $x_2(k)$, · · · , $x_n(k)$) and ($x^*_1(k)$, $x^*_2(k)$, · · · , $x^*_n(k)$) of system (1.1) with initial condition (1.2), we have

$$\lim_{k\to+\infty}(x_i^*(k)-x_i(k))=0,\quad i=1,2,\cdots,n.$$

Proof. Firstly, let

$$V_{i1}(k)=\big|\ln x_i(k)-\ln x_i^*(k)\big|.$$

From (4.1), we have that for k ≥ 2τ,

$$\Big|\ln\frac{x_i(k+1)}{x_i^*(k+1)}\Big|\leq\Big|\ln\frac{x_i(k)}{x_i^*(k)}-b_i(k)[x_i(k)-x_i^*(k)]\Big|+\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)}{d_{ij}(k)}|x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})|$$
$$+b_i(k)\sum_{s=k-\sigma_i}^{k-1}\Big\{|x_i(s)-x_i^*(s)|A_i(s)\big[a_i(s)+b_i(s)|x_i^*(s-\sigma_i)|+\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(s)}{d_{ij}(s)}|x_j^*(s-\tau_{ij})|\big]$$
$$+|x_i(s)|B_i(s)\Big[\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(s)}{d_{ij}(s)}|x_j(s-\tau_{ij})-x_j^*(s-\tau_{ij})|+b_i(s)|x_i(s-\sigma_i)-x_i^*(s-\sigma_i)|\Big]\Big\}. \tag{4.8}$$

Since

$$x_i(k)-x_i^*(k)=e^{\ln x_i(k)}-e^{\ln x_i^*(k)}=\xi_i(k)\ln(x_i(k)/x_i^*(k)),\quad i=1,2,\cdots,n,$$

where $\xi_i(k)$ lies between $x_i(k)$ and $x^*_i(k)$, i = 1, 2, · · · , n, it follows that

$$\big|\ln(x_i(k)/x_i^*(k))-b_i(k)[x_i(k)-x_i^*(k)]\big|$$
$$=\big|\ln(x_i(k)/x_i^*(k))-b_i(k)\xi_i(k)\ln(x_i(k)/x_i^*(k))\big|$$
$$=\big|\ln(x_i(k)/x_i^*(k))\big|-\Big(\frac{1}{\xi_i(k)}-\Big|\frac{1}{\xi_i(k)}-b_i(k)\Big|\Big)|x_i(k)-x_i^*(k)|. \tag{4.9}$$

By Theorem 3.1, there are constants $M_i > 0$, and a positive integer k_0 such that for $k > k_0$, $0 < x_i(k)$, $x^*_i(k) \leq M_i$, i = 1, 2, · · · , n. Then from (4.8) and (4.9) we can obtain that for $k \geq k_0 + 2\tau$,

$$\triangle V_{i1}(k)\leq-\Big(\frac{1}{\xi_i(k)}-\Big|\frac{1}{\xi_i(k)}-b_i(k)\Big|\Big)|x_i(k)-x_i^*(k)|+\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)}{d_{ij}(k)}|x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})|$$
$$+b_i(k)\sum_{s=k-\sigma_i}^{k-1}\Big\{A_i(s)\big[a_i(s)+M_ib_i(s)+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}(s)}{d_{ij}(s)}\big]|x_i(s)-x_i^*(s)|$$
$$+M_iB_i(s)\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(s)}{d_{ij}(s)}|x_j(s-\tau_{ij})-x_j^*(s-\tau_{ij})|$$
$$+M_iB_i(s)b_i(s)\big|x_i(s-\sigma_i)-x_i^*(s-\sigma_i)\big|\Big\}. \tag{4.10}$$

Secondly, let

$$V_{i2}(k)=\sum_{j=1,j\neq i}^{n}\sum_{s=k-\tau_{ij}}^{k-1}\frac{c_{ij}(s+\tau_{ij})}{d_{ij}(s+\tau_{ij})}|x_j(s)-x_j^*(s)|$$
$$+\sum_{s=k}^{k-1+\sigma_i}b_i(s)\sum_{u=s-\sigma_i}^{k-1}\Big\{A_i(u)\big[a_i(u)+M_ib_i(u)+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}(u)}{d_{ij}(u)}\big]|x_i(u)-x_i^*(u)|$$
$$+M_iB_i(u)\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(u)}{d_{ij}(u)}|x_j(u-\tau_{ij})-x_j^*(u-\tau_{ij})|$$
$$+M_iB_i(u)b_i(u)\big|x_i(u-\sigma_i)-x_i^*(u-\sigma_i)\big|\Big\}. \tag{4.11}$$

By a simple calculation, we can obtain

$$\triangle V_{i2}(k)=\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k+\tau_{ij})}{d_{ij}(k+\tau_{ij})}\big|x_j(k)-x_j^*(k)\big|-\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)}{d_{ij}(k)}\big|x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})\big|$$
$$+\Big\{A_i(k)\big[a_i(k)+M_ib_i(k)+\sum_{j=1,j\neq i}^{n}M_j\frac{c_{ij}(k)}{d_{ij}(k)}\big]|x_i(k)-x_i^*(k)|$$
$$+M_iB_i(k)\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)}{d_{ij}(k)}|x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})|$$
$$+M_iB_i(k)b_i(k)\big|x_i(k-\sigma_i)-x_i^*(k-\sigma_i)\big|\Big\}\sum_{s=k+1}^{k+\sigma_i}b_i(s)$$
$$-b_i(k)\sum_{u=k-\sigma_i}^{k-1}\Big\{A_i(u)\big[a_i(u)+M_ib_i(u)+\sum_{j=1,j\neq i}^{n}M_j\frac{c_{ij}(u)}{d_{ij}(u)}\big]|x_i(u)-x_i^*(u)|$$
$$+M_iB_i(u)\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(u)}{d_{ij}(u)}|x_j(u-\tau_{ij})-x_j^*(u-\tau_{ij})|$$
$$+M_iB_i(u)b_i(u)\big|x_i(u-\sigma_i)-x_i^*(u-\sigma_i)\big|\Big\}. \tag{4.12}$$

Thirdly, let

$$V_{i3}(k)=M_i\sum_{j=1,j\neq i}^{n}\sum_{l=k-\tau_{ij}}^{k-1}\sum_{s=l+\tau_{ij}+1}^{l+\tau_{ij}+\sigma_i}b_i(s)B_i(l+\tau_{ij})\frac{c_{ij}(l+\tau_{ij})}{d_{ij}(l+\tau_{ij})}|x_j(l)-x_j^*(l)|$$
$$+M_i\sum_{l=k-\sigma_i}^{k-1}\sum_{s=l+\sigma_i+1}^{l+2\sigma_i}b_i(s)B_i(l+\sigma_i)b_i(l+\sigma_i)\big|x_i(l)-x_i^*(l)\big|.$$

Then we can derive

$$\triangle V_{i3}(k)=M_i\sum_{j=1,j\neq i}^{n}B_i(k+\tau_{ij})\frac{c_{ij}(k+\tau_{ij})}{d_{ij}(k+\tau_{ij})}|x_j(k)-x_j^*(k)|\sum_{s=k+\tau_{ij}+1}^{k+\tau_{ij}+\sigma_i}b_i(s)$$
$$-M_iB_i(k)\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(k)}{d_{ij}(k)}|x_j(k-\tau_{ij})-x_j^*(k-\tau_{ij})|\sum_{s=k+1}^{k+\sigma_i}b_i(s)$$
$$+M_iB_i(k+\sigma_i)b_i(k+\sigma_i)\big|x_i(k)-x_i^*(k)\big|\sum_{s=k+\sigma_i+1}^{k+2\sigma_i}b_i(s)$$
$$-M_iB_i(k)b_i(k)\big|x_i(k-\sigma_i)-x_i^*(k-\sigma_i)\big|\sum_{s=k+1}^{k+\sigma_i}b_i(s). \tag{4.13}$$

Now we set

$$V_i(k) = V_{i1}(k) + V_{i2}(k) + V_{i3}(k),\ i = 1, 2, \cdots, n.$$

Then from (4.8)-(4.13), we have that for $k \ge k_0+2\tau$,

$$\triangle V_i(k) \le -\Big(\frac{1}{\xi_i(k)} - \Big|\frac{1}{\xi_i(k)} - b_i(k)\Big|\Big)\big|x_i(k) - x_i^*(k)\big| + \sum_{j=1,j\ne i}^{n} \frac{c_{ij}(k+\tau_{ij})}{d_{ij}(k+\tau_{ij})}\big|x_j(k) - x_j^*(k)\big|$$
$$+ A_i(k)\big[a_i(k) + M_i b_i(k) + \sum_{j=1,j\ne i}^{n} \frac{M_j c_{ij}(k)}{d_{ij}(k)}\big] \sum_{s=k+1}^{k+\sigma_i} b_i(s)\big|x_i(k) - x_i^*(k)\big|$$
$$+ M_i \sum_{j=1,j\ne i}^{n} B_i(k+\tau_{ij}) \frac{c_{ij}(k+\tau_{ij})}{d_{ij}(k+\tau_{ij})} \sum_{s=k+\tau_{ij}+1}^{k+\tau_{ij}+\sigma_i} b_i(s)\big|x_j(k) - x_j^*(k)\big|$$
$$+ M_i B_i(k+\sigma_i) b_i(k+\sigma_i) \sum_{s=k+\sigma_i+1}^{k+2\sigma_i} b_i(s)\big|x_i(k) - x_i^*(k)\big|.$$

Now we define a Lyapunov-like discrete functional $V(k)$ by

$$V(k) = \sum_{i=1}^{n} \beta_i V_i(k).$$

It is easy to see that $V(k_0+2\tau)<+\infty$. Calculating the difference of $V(k)$ along the solution of system (1.1) with initial condition (1.2), we have that for $k \ge k_0+2\tau$,

$$\triangle V(k) \le -\sum_{i=1}^{n}\Big\{\beta_i\Big[\Big(\frac{1}{\xi_i(k)} - \Big|\frac{1}{\xi_i(k)} - b_i(k)\Big|\Big) - M_i B_i(k+\sigma_i) b_i(k+\sigma_i)\sum_{s=k+\sigma_i+1}^{k+2\sigma_i} b_i(s)$$
$$- A_i(k)\big[a_i(k) + M_i b_i(k) + \sum_{j=1,j\ne i}^{n}\frac{M_j c_{ij}(k)}{d_{ij}(k)}\big]\sum_{s=k+1}^{k+\sigma_i} b_i(s)\Big]$$
$$- \sum_{j=1,j\ne i}^{n} \frac{\beta_j c_{ji}(k+\tau_{ji})}{d_{ji}(k+\tau_{ji})} - \sum_{j=1,j\ne i}^{n} \beta_j M_j B_j(k+\tau_{ji}) \frac{c_{ji}(k+\tau_{ji})}{d_{ji}(k+\tau_{ji})} \sum_{s=k+\tau_{ji}+1}^{k+\tau_{ji}+\sigma_j} b_j(s)\Big\}\big|x_i(k) - x_i^*(k)\big|$$
$$\le -\sum_{i=1}^{n}\Big\{\beta_i\Big[\min\{b_i^l, \frac{2}{M_i} - b_i^u\} - \sigma_i M_i (b_i^u)^2 B_i^u - \sigma_i b_i^u A_i^u\Big(a_i^u + b_i^u M_i + \sum_{j=1,j\ne i}^{n}\frac{M_j c_{ij}^u}{d_{ij}^l}\Big)\Big]$$
$$- \sum_{j=1,j\ne i}^{n} \beta_j \frac{c_{ij}^u}{d_{ij}^l}(1 + \sigma_j M_j b_j^u B_j^u)\Big\}\big|x_i(k) - x_i^*(k)\big|$$
$$= -\sum_{i=1}^{n}\big(\beta_i E_i - \sum_{j=1,j\ne i}^{n}\beta_j F_{ij}\big)\big|x_i(k) - x_i^*(k)\big|$$
$$\le -\eta\sum_{i=1}^{n}\big|x_i(k) - x_i^*(k)\big|,$$

where E_i and F_{ij} are defined by (4.7).

Then we have that

$$\sum_{p=k_0+2\tau}^{k}[V(p+1) - V(p)] \le -\eta \sum_{p=k_0+2\tau}^{k}\sum_{i=1}^{n}\big|x_i(p) - x_i^*(p)\big|,$$

which implies

$$V(k+1) + \eta \sum_{p=k_0+2\tau}^{k}\sum_{i=1}^{n}\big|x_i(p) - x_i^*(p)\big| \le V(k_0+2\tau).$$

That is

$$\sum_{p=k_0+2\tau}^{k}\sum_{i=1}^{n}\big|x_i(p) - x_i^*(p)\big| \le \frac{V(k_0+2\tau)}{\eta},$$

and then

$$\sum_{k=k_0+2\tau}^{+\infty}\sum_{i=1}^{n}\big|x_i(k) - x_i^*(k)\big| \le \frac{V(k_0+2\tau)}{\eta} < +\infty,$$

which means that $\lim_{k\to+\infty}\sum_{i=1}^{n}\big|x_i(k) - x_i^*(k)\big| = 0$, that is

$$\lim_{k\to+\infty}(x_i(k) - x_i^*(k)) = 0,\quad i = 1, 2, \cdots, n.$$

It means that $(x_1(k), x_2(k), \cdots, x_n(k))$ is globally attractive. This completes the proof of Theorem 4.1.

V. ALMOST PERIODIC SOLUTION

In this section, we will study the existence of a globally attractive almost periodic sequence solution of system (1.1) with initial condition (1.2) by means of an almost periodic functional hull theory and constructing a suitable Lyapunov function, and obtain the sufficient conditions.

Let $\{\delta_m\}$ be any integer valued sequence such that $\delta_m \to\infty$ as $m\to\infty$. According to Lemma 2.1, taking a subsequence if necessary, we have

$a_i(k+\delta_m) \to a_i^*(k), b_i(k+\delta_m) \to b_i^*(k), c_{ij}(k+\delta_m) \to c_{ij}^*(k), d_{ij}(k+\delta_m) \to d_{ij}^*(k),\ i, j = 1, 2, \cdots, n, j \ne i$, as $m\to\infty$ for $k \in \mathbf{Z}$.

Then we get a hull equation of system (1.1) as follows:

$$x_i(k+1) = x_i(k)\exp\Big\{a_i^*(k) - b_i^*(k)x_i(k-\sigma_i) + \sum_{j=1,j\ne i}^{n} c_{ij}^*(k)\frac{x_j(k-\tau_{ij})}{d_{ij}^*(k) + x_j(k-\tau_{ij})}\Big\},\quad i = 1, 2, \cdots, n. \tag{5.1}$$

By the almost periodic theory, we can conclude that if system (1.1) satisfies (4.6), then the hull equation (5.1) of system (1.1) also satisfies (4.6).

By Theorem 3.4 in [26], we can easily obtain the lemma as follows.

Lemma 5.1 If each hull equation of system (1.1) has a unique strictly positive solution, then the almost periodic difference system (1.1) has a unique strictly positive almost periodic solution.

Theorem 5.1 If the almost periodic difference system (1.1) satisfies (4.6), then the almost periodic difference system (1.1) admits a unique strictly positive almost periodic solution, which is globally attractive.

Proof. By Lemma 5.1, we only need to prove that each hull equation of system (1.1) has a unique globally attractive almost periodic sequence solution; hence we firstly prove that each hull equation of system (1.1) has at least one strictly positive solution (the existence), and then we prove that each hull equation of system (1.1) has a unique strictly positive solution (the uniqueness).

Now we prove the existence of a strictly positive solution of any hull equation (5.1). By the almost periodicity of $\{a^*_i(k)\}$, $\{b^*_i(k)\}$, $\{c^*_{ij}(k)\}$ and $\{d^*_{ij}(k)\}$, there exists an integer valued sequence $\{\tau_m\}$ with $\tau_m\to\infty$ as $m\to\infty$ such that $a^*_i(k+\tau_m) \to a^*_i(k)$, $b^*_i(k+\tau_m) \to b^*_i(k)$, $c^*_{ij}(k+\tau_m) \to c^*_{ij}(k)$, $d^*_{ij}(k+\tau_m) \to d^*_{ij}(k)$, as $m\to\infty$ for $k\in Z$. Suppose that $X(k) = (x_1(k), x_2(k), \cdots, x_n(k))$ is any solution of hull equation (5.1). By the proof of Lemma 2.2 and 2.3, we have

$$m_i \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_i,\quad i = 1, 2, \cdots, n. \tag{5.2}$$

And also

$$0 < \inf_{k\in\mathbf{Z}^+} x_i(k) \le \sup_{k\in\mathbf{Z}^+} x_i(k) < \infty,\qquad i = 1, 2, \cdots, n.$$

Let ε be an arbitrary small positive number. It from (5.2) that there exists a positive integer k_0 such that $m_i-\varepsilon\le x_i(k)\le M_i+\varepsilon$, $k\ge k_0$, $i = 1, 2, \cdots, n$. Write $X_m(k) = X(k+\tau_m) = (x_{1m}(k), x_{2m}(k), \cdots, x_{nm}(k))$, for all $k \ge k_0 + \tau - \tau_m, m \in Z^+$. We claim that there exists a sequence $\{y_i(k)\}$, and a subsequence of $\{\tau_k\}$, we still denote by $\{\tau_k\}$ such that $x_{im}(k) \to y_i(k)$, uniformly in k on any finite subset B of Z as $m\to\infty$, where $B = \{a_1, a_2, \ldots, a_p\}$, $a_h\in Z(h = 1, 2, \ldots, p)$ and p is a finite number.

In fact, for any finite subset $B\subset Z$, when m is large enough, $\tau_m + a_h - \tau>k_0$, $h = 1, 2, \ldots, p$. So

$$m_i - \varepsilon \le x_i(k+\tau_m) \le M_i + \varepsilon,\quad i = 1, 2, \cdots, n,$$

that is, $\{x_i(k+\tau_m)\}$ are uniformly bounded for large enough m.

Now, for $a_1\in B$, we can choose a subsequence $\{\tau^{(1)}_m\}$ of $\{\tau_m\}$ such that $\{x_i(a_1+\tau^{(1)}_m)\}$ uniformly converges on Z^+ for m large enough.

Similarly, for $a_2\in B$, we can choose a subsequence $\{\tau^{(2)}_m\}$ of $\{\tau^{(1)}_m\}$ such that $\{x_i(a_2+\tau^{(2)}_m)\}$ uniformly converges on Z^+ for m large enough.

Repeating this procedure, for $a_p \in B$, we can choose a subsequence $\{\tau^{(p)}_m\}$ of $\{\tau^{(p-1)}_m\}$ such that $\{x_i(a_p+\tau^{(p)}_m)\}$ uniformly converges on Z^+ for m large enough.

Now pick the sequence $\{\tau^{(p)}_m\}$ which is a subsequence of $\{\tau_m\}$, we still denote it as $\{\tau_m\}$, then for all $k\in B$, we have $x_i(k+\tau_m)\to y_i(k)$ uniformly in $k\in B$, as $m\to\infty$.

By the arbitrary of B, the conclusion is valid.

Combined with

$$x_{im}(k+1) = x_{im}(k)\exp\left\{a_i^*(k+\tau_m) - b_i^*(k+\tau_m)x_{im}(k-\sigma_i) + \sum_{j=1,j\neq i}^{n} c_{ij}^*(k+\tau_m)\frac{x_{jm}(k-\tau_{ij})}{d_{ij}^*(k+\tau_m)+x_{jm}(k-\tau_{ij})}\right\}, \quad i=1,2,\cdots,n,$$

gives

$$y_i(k+1) = y_i(k)\exp\left\{a_i^*(k) - b_i^*(k)y_i(k-\sigma_i) + \sum_{j=1,j\neq i}^{n} c_{ij}^*(k)\frac{y_j(k-\tau_{ij})}{d_{ij}^*(k)+y_j(k-\tau_{ij})}\right\}, \quad i=1,2,\cdots,n.$$

We can easily see that $Y(k) = (y_1(k), y_2(k), \cdots, y_n(k))$ is a solution of hull equation (5.1) and $m_i-\varepsilon\le y_i(k)\le M_i+\varepsilon$, $i = 1, 2, \cdots, n$, for $k\in \mathbf{Z}$. Since ε is an arbitrary small positive number, it follows that $m_i\le y_i(k)\le M_i$, $i = 1, 2, \cdots, n$, for $k\in \mathbf{Z}$, that is

$$0<\inf_{k\in\mathbf{Z}} y_i(k)\le \sup_{k\in\mathbf{Z}} y_i(k)<\infty, \quad i=1,2,\cdots,n.$$

Hence each hull equation of almost periodic difference system (1.1) has at least one strictly positive solution.

Now we prove the uniqueness of the strictly positive solution of each hull equation (5.1). Suppose that the hull equation (5.1) has two arbitrary strictly positive solutions $(x^*_1(k), x^*_2(k), \cdots, x^*_n(k))$ and $(y^*_1(k), y^*_2(k),\cdots, y^*_n(k))$. Like in the proof of Theorem 4.1, we construct a Lyapunov functional

$$V^*(k) = \sum_{i=1}^{n}\beta_i\Big(V_{i1}^*(k)+V_{i2}^*(k)+V_{i3}^*(k)\Big), \quad k\in\mathbf{Z}, \tag{5.3}$$

where

$$V_{i1}^*(k) = |\ln x_i^*(k) - \ln y_i^*(k)|,$$

$$\begin{aligned}V_{i2}^*(k) = &\sum_{j=1,j\neq i}^{n}\sum_{s=k-\tau_{ij}}^{k-1}\frac{c_{ij}(s+\tau_{ij})}{d_{ij}(s+\tau_{ij})}|x_j^*(s)-y_j^*(s)|\\ &+\sum_{s=k}^{k-1+\sigma_i} b_i(s)\sum_{u=s-\sigma_i}^{k-1}\Big\{A_i(u)\big[a_i(u)+M_ib_i(u)+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}(u)}{d_{ij}(u)}\big]|x_i^*(u)-y_i^*(u)|\\ &+M_iB_i(u)\sum_{j=1,j\neq i}^{n}\frac{c_{ij}(u)}{d_{ij}(u)}|x_j^*(u-\tau_{ij})-y_j^*(u-\tau_{ij})|\\ &+M_iB_i(u)b_i(u)\big|x_i^*(u-\sigma_i)-y_i^*(u-\sigma_i)\big|\Big\},\end{aligned}$$

$$\begin{aligned}V_{i3}^*(k) = &M_i\sum_{j=1,j\neq i}^{n}\sum_{l=k-\tau_{ij}}^{k-1}\sum_{s=l+\tau_{ij}+1}^{l+\tau_{ij}+\sigma_i} b_i(s)B_i(l+\tau_{ij})\frac{c_{ij}(l+\tau_{ij})}{d_{ij}(l+\tau_{ij})}|x_j^*(l)-y_j^*(l)|\\ &+M_i\sum_{l=k-\sigma_i}^{k-1}\sum_{s=l+\sigma_i+1}^{l+2\sigma_i} b_i(s)B_i(l+\sigma_i)b_i(l+\sigma_i)\big|x_i^*(l)-y_i^*(l)\big|.\end{aligned}$$

Calculating the difference of $V^*(k)$ along the solution of the hull equation (5.1), like in the discussion of (4.14), one has

$$\Delta V^*(k)\le -\eta\sum_{i=1}^{n}|x_i^*(k)-y_i^*(k)|, \quad k\in\mathbf{Z}. \tag{5.4}$$

From (5.4), we can see that $V^*(k)$ is a non-increasing function on Z. Summing both sides of the above inequalities from k to 0, we have

$$\eta\sum_{q=k}^{0}\sum_{i=1}^{n}|x_i^*(q)-y_i^*(q)|\le V^*(0)-V^*(k+1), \quad k<0.$$

Note that $V^*(k)$ is bounded. Hence we have

$$\sum_{q=-\infty}^{0}\sum_{i=1}^{n}|x_i^*(q)-y_i^*(q)|<+\infty,$$

which implies that

$$\lim_{k\to-\infty}|x_i^*(k)-y_i^*(k)|=0, \quad i=1,2,\cdots,n. \tag{5.5}$$

Define $Q=\sum_{i=1}^{n}\beta_iQ_i$, where

$$\begin{aligned}Q_i = &\frac{1}{m_i}+\sum_{j=1,j\neq i}^{n}\frac{\tau_{ij}c_{ij}^u}{d_{ij}^l}+\sigma_i^2b_i^u\big[A_i^u(a_i^u+M_ib_i^u+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}^u}{d_{ij}^l})+M_iB_i^u(\sum_{j=1,j\neq i}^{n}\frac{c_{ij}^u}{d_{ij}^l}+b_i^u)\big]\\ &+M_i\sigma_ib_i^uB_i^u(\sum_{j=1,j\neq i}^{n}\frac{\tau_{ij}c_{ij}^u}{d_{ij}^l}+\sigma_ib_i^u), \quad i=1,2,\cdots,n.\end{aligned}$$

Let ε be an arbitrary small positive number. It follows from (5.5) that there exists a positive integer $K > 0$ such that

$$|x_i^*(k)-y_i^*(k)|<\frac{\varepsilon}{Q}, k<-K, i=1,2,\cdots,n.$$

Therefore, for $k<-K$, $i = 1, 2, \cdots, n$

$$V_{i1}^*(k)\le\frac{1}{m_i}|x_i^*(k)-y_i^*(k)|\le\frac{1}{m_i}\frac{\varepsilon}{Q},$$

$$\begin{aligned}V_{i2}^*(k)\le &\sum_{j=1,j\neq i}^{n}\frac{\tau_{ij}c_{ij}^u}{d_{ij}^l}\max_{p\le k}|x_j^*(p)-y_j^*(p)|\\ &+\sigma_i^2b_i^u\Big[A_i^u(a_i^u+M_ib_i^u+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}^u}{d_{ij}^l})\max_{p\le k}|x_i^*(p)-y_i^*(p)|\\ &+M_iB_i^u\sum_{j=1,j\neq i}^{n}\frac{c_{ij}^u}{d_{ij}^l}\max_{p\le k}|x_j^*(p)-y_j^*(p)|+M_iB_i^ub_i^u\max_{p\le k}|x_i^*(p)-y_i^*(p)|\Big]\\ \le &\Big\{\sum_{j=1,j\neq i}^{n}\frac{\tau_{ij}c_{ij}^u}{d_{ij}^l}+\sigma_i^2b_i^u\big[A_i^u(a_i^u+M_ib_i^u+\sum_{j=1,j\neq i}^{n}\frac{M_jc_{ij}^u}{d_{ij}^l})+M_iB_i^u(\sum_{j=1,j\neq i}^{n}\frac{c_{ij}^u}{d_{ij}^l}+b_i^u)\big]\Big\}\frac{\varepsilon}{Q},\end{aligned}$$

$$\begin{aligned}V_{i3}^*(k)\le &\sigma_iM_ib_i^uB_i^u\sum_{j=1,j\neq i}^{n}\frac{\tau_{ij}c_{ij}^u}{d_{ij}^l}\max_{p\le k}|x_j^*(p)-y_j^*(p)|+M_i(\sigma_i)^2(b_i^u)^2B_i^u\max_{p\le k}|x_i^*(p)-y_i^*(p)|\\ \le &M_i\sigma_ib_i^uB_i^u(\sum_{j=1,j\neq i}^{n}\frac{\tau_{ij}c_{ij}^u}{d_{ij}^l}+\sigma_ib_i^u)\frac{\varepsilon}{Q}.\end{aligned}$$

It follows from (5.3) and above inequalities that

$$V^*(k)\le\sum_{i=1}^{n}\beta_iQ_i\frac{\varepsilon}{Q}=\varepsilon,\ k<-K,$$

so $\lim_{k\to-\infty}V^*(k)=0$. Note that $V^*(k)$ is a non-increasing function on Z, and then $V^*(k)\equiv 0$. That is $x^*_i(k) = y^*_i(k)$, $i = 1, 2, \cdots, n$, for all $k\in\mathbf{Z}$, Therefore, each hull equation of system (1.1) has a unique strictly positive solution.

In view of the above discussion, any hull equation of system (1.1) has a unique strictly positive solution. By Lemma 2.2-2.3 and Theorem 4.1, the almost periodic difference system (1.1) has a unique strictly positive almost periodic solution which is globally attractive. The proof is completed.

VI AN EXAMPLE AND NUMERICAL SIMULATION

In this section, we give the following example to check the feasibility of our result.

Example Consider the following almost periodic discrete Lotka-Volterra mutualism model with delays:

$$\begin{aligned}x_1(k+1) = x_1(k)\exp\Big\{&0.025+0.005\sin(\sqrt{2}k)-(1.0075-0.0025\cos(\sqrt{3}k))x_1(k-1)\\ &+(0.03-0.005\cos(\sqrt{2}k))\frac{x_2(k-2)}{4.02+0.005\sin(\sqrt{2}k)+x_2(k-2)}\\ &+(0.02+0.006\sin(\sqrt{3}k))\frac{x_3(k-3)}{4.03+0.005\cos(\sqrt{3}k)+x_3(k-3)}\Big\},\end{aligned}$$

$$\begin{aligned}x_2(k+1) = x_2(k)\exp\Big\{&0.035+0.005\cos(\sqrt{3}k)-(1.0025+0.0015\sin(\sqrt{2}k))x_2(k-2)\\ &+(0.02-0.004\sin(\sqrt{2}k))\frac{x_1(k-1)}{5.03+0.006\cos(\sqrt{3}k)+x_1(k-1)}\\ &+(0.025+0.005\cos(\sqrt{5}k))\frac{x_3(k-2)}{5.04+0.01\sin(\sqrt{2}k)+x_3(k-2)}\Big\},\end{aligned} \tag{6.1}$$

$$x_3(k+1) = x_3(k)\exp\Big\{0.026 - 0.006\sin(\sqrt{5}k) - (1.0035 + 0.0025\cos(\sqrt{2}k))x_3(k-1)$$
$$+(0.03 + 0.005\cos(\sqrt{3}k))\frac{x_1(k-2)}{4.08 - 0.004\sin(\sqrt{3}k) + x_1(k-2)}$$
$$+(0.02 - 0.006\sin(\sqrt{5}k))\frac{x_2(k-3)}{4.06 + 0.008\cos(\sqrt{2}k) + x_2(k-3)}\Big\}.$$

By simple computation, we derive

$$M_1 \approx 0.0261,\quad M_2 \approx 0.0392,\quad M_3 \approx 0.0283,$$
$$E_1 \approx 0.0347,\quad F_{12} \approx 0.0177,\quad F_{13} \approx 0.0152,$$
$$E_2 \approx 0.0358,\quad F_{21} \approx 0.0136,\quad F_{23} \approx 0.0168,$$
$$E_3 \approx 0.0407,\quad F_{31} \approx 0.0145,\quad F_{32} \approx 0.0183.$$

Then

$$E_1 - F_{12} - F_{13} \approx 0.0018 > 0.001,$$
$$E_2 - F_{21} - F_{23} \approx 0.0024 > 0.001,$$
$$E_3 - F_{31} - F_{32} \approx 0.0021 > 0.001.$$

Also it is easy to see that the condition (4.6) is verified. Therefore, system (6.1) has a unique strictly positive almost periodic solution which is globally attractive. Our numerical simulations support our results (see Figs.1-3).

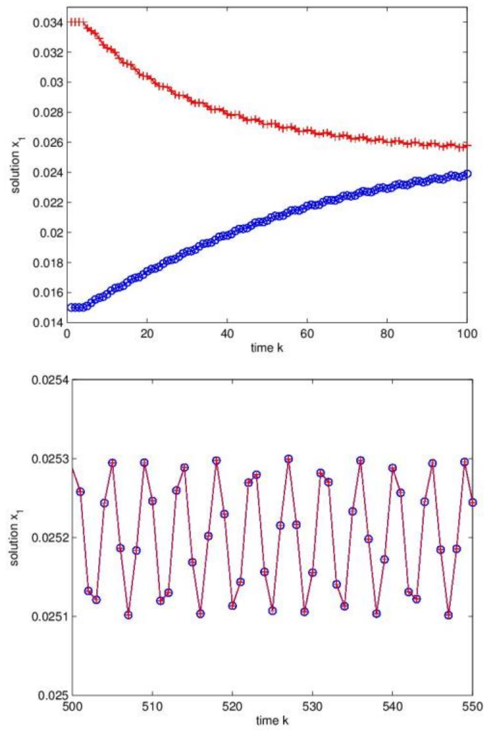
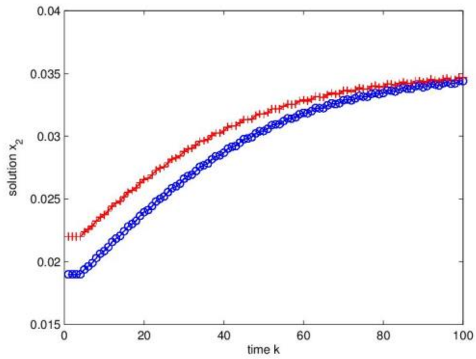


FIGURE1: Dynamic behavior of $x_1(k)$ of system (6.1) with the initial conditions $(x_1(k), x_2(k), x_3(k))$ = (0.015, 0.019, 0.011) and (0.034, 0.022, 0.028), k = 1, 2, 3, 4 for k ∈ [1, 100] and k ∈ [500, 550], respectively.



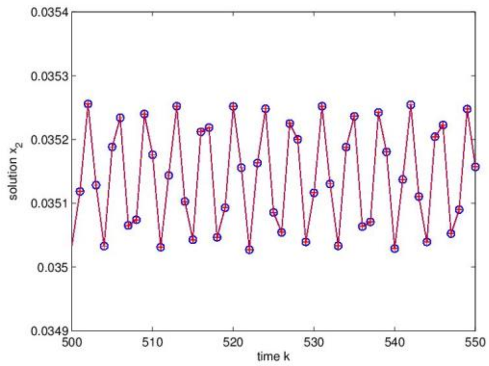


FIGURE2: Dynamic behavior of $x_2(k)$ of system (6.1) with the initial conditions $(x_1(k), x_2(k), x_3(k))$ = (0.015, 0.019, 0.011) and (0.034, 0.022, 0.028), k = 1, 2, 3, 4 for k ∈ [1, 100] and k ∈ [500, 550], respectively.

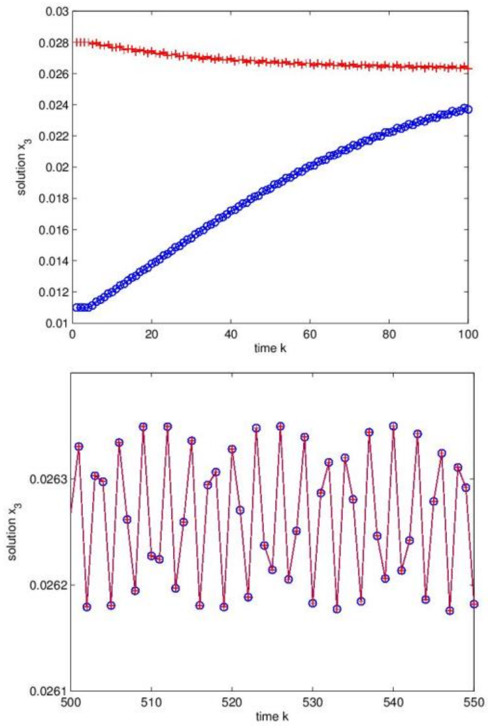


FIGURE3: Dynamic behavior of $x_3(k)$ of system (6.1) with the initial conditions $(x_1(k), x_2(k), x_3(k))$ = (0.015, 0.019, 0.011) and (0.034, 0.022, 0.028), k = 1, 2, 3, 4 for k ∈ [1, 100] and k ∈ [500, 550], respectively.

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