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The Change Effect for Boundary Conditions to Minimize Error Bound in Lacunary Interpolation by Forth Spline Function

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***Abstract*—The researcher interested in spline function because it is necessary function in interpolation process, by putting a certain condition for spline function from forth degree which is as follows**

$S_n''(x_0) = 0, S_n''(x_{2m}) = 0$, **then the aim of this research is to change the previous boundary conditions which is as** $S_n''(x_0) = f''(x_0), S_n''(x_{2m}) = f''(x_{2m})$ **to get minimum error for spline function from forth degree.**

***Index Terms*— Spline function ; Boundary Conditions.**

I. INTRODUCTION

Spline functions appears in the late middle of twentieth century, According to Schoenbery [5] , the interest in Spline functions is due to the fact that , Spline function are good tool for the numerical approximation of functions and that they suggest new ,challenging and rewarding problems . For more information about a Spline function , one is referred to Alberg ,Milson and walsh [1] .Lacunary interpolation by Spline appears whenever observation gives scattered or irregular information about a function and its derivatives , but with out Hermite condition . Mathematically , in the problem of interpolation given data $a_{i,j}$ by polynomial $p_n(x)$ of degree of most n satisfying :

$$p_n^{(j)}(x_i) = a_{i,j} \quad ,i = 1,2,3,...,n \quad , \; j = 0,1,2,......., m$$

we have Hermit interpolation if for i , the order j of derivatives in form unbroken sequence . If some of the sequence are broken , we have lacunary interpolation . In another communications we shall give the applicationsof Spline functions obtaining the approximation solution of boundary value problem , for more about applications of Spline functions , see [2,4] . For description our problem , let

$\Delta : 0 = x_o \le x_2 \le x_3 \le ..... \le x_{2m} = 1$ by a uniform partition of the interval [0,1] with $x_i = \dfrac{i}{2m}$ $, i = 0,2,....,2m$ and n=2m+1 . We define the class of Spline function $S_P(4,3,n)$ as follows :

any element $S_\Delta(x) \in S_P(4,3,n)$ if the following condition satisfied :

$$\left.\begin{array}{l}(i)\, S_\Delta(x) \in C^3[0,1] \\ (ii)\, S_\Delta(x)\, is\ polynomial\ of\ degree\ four\ in\ each\, [x_{2i}, x_{2i+2}] \\ i = 0,1,...,m-1\end{array}\right] \quad (1)$$

In this paper we prove the following :

Theorem 1:

Given arbitrary number $f(x_{2i})$, $f(t_{2i})$ i=0,1,2…,n-1 and $f^{(2)}(x_\circ), f^{(2)}(x_{2m})$ there exists a unique spline $S_n(x) \in S_p(4,3,n)$ such that

$$\left.\begin{array}{ll}S_n(x_{2i}) = f(x_{2i}) & i = 0,1,2.....,m \\ S_n(t_{2i}) = f(t_{2i}) & i = 0,1,2....,m-1 \\ S_n^{(2)}(x_0) = f^{(2)}(x_0), & S_n^{(2)}(x_{2m}) = f^{(2)}(x_{2m})\end{array}\right] \quad (2)$$

Theorem 2:

Let $f \in C^4[0,1]$ and $S_n \in Sp(4,3,n)$ be a unique spline satisfying the condition of (theorem 1.1) , then

$$\left\| S_n^{(r)}(x) - f^{(r)}(x) \right\| \le 5.6404257 m^{r-4} w(f, \frac{1}{m}) + m^{r-4} \left\| f^{(4)} \right\| \quad ; r = 0,1,2,3$$

where

$$w(f^{(4)}, \frac{1}{m}) = \max \left\| f^{(4)}(x) - f^{(4)}(y) \right\| : |x-y| \le h, \forall x, y \in [0,1]$$

$$\left\| f^{(4)} \right\| = \max \left\{ \left| f^{(4)}(x) \right| \right\} : \; 0 \le x \le 1$$

II. TECHNICAL PRELIMINARIES:

If p(x) is a polynomial of degree Four on [0,1] (because we want to construct a spline function of degree four) then we have :

$$P(x) = P(0)A_0(x) + P(\frac{1}{3})A_1(x) + P(1)A_2(x) + P''(0)A_3(x) + P''(1)A_4(x)$$

……..(3)

Where

$$\left.\begin{aligned}
A_0(x) &= \frac{1}{11}[-27x^4 + 54x^3 - 38x + 11] \\
A_1(x) &= \frac{1}{22}[81x^4 - 162x^3 + 81x] \\
A_2(x) &= \frac{1}{22}[-27x^4 + 54x^3 - 5x] \\
A_3(x) &= \frac{1}{66}[15x^4 - 41x^3 + 33x^2 - 7x] \\
A_4(x) &= \frac{1}{66}[12x^4 - 13x^3 + x]
\end{aligned}\right] \quad ....(4)$$

In the subsequent section we need the following values for $f \in C^4[0,1]$ we have the following expansions:

$$f(x_{2i+2}) = f(x_{2i}) + 2hf^{/}(x_{2i}) + 2h^2 f^{//}(x_{2i}) + \frac{4}{3}h^3 f^{///}(x_{2i}) + \frac{2}{3}h^4 f^{(4)}(\lambda_{1,2i})$$

$$x_{2i} \le \lambda_{1,2i} \le x_{2i+2}$$

$$f(x_{2i-2}) = f(x_{2i}) - 2hf^{/}(x_{2i}) + 2h^2 f^{//}(x_{2i}) - \frac{4}{3}h^3 f^{///}(x_{2i}) + \frac{2}{3}h^4 f^{(4)}(\lambda_{2,2i})$$

$$x_{2i-2} \le \lambda_{2,2i} \le x_{2i}$$

$$f(t_{2i}) = f(x_{2i}) + \frac{2}{3}hf^{/}(x_{2i}) + \frac{2}{9}h^2 f^{//}(x_{2i}) + \frac{4}{81}h^3 f^{///}(x_{2i}) + \frac{2}{243}h^4 f^{(4)}(\lambda_{3,2i})$$

$$x_{2i} \le \lambda_{3,2i} \le t_{2i}$$

$$f(t_{2i-2}) = f(x_{2i}) - \frac{4}{3}hf^{/}(x_{2i}) + \frac{8}{9}h^2 f^{//}(x_{2i}) - \frac{32}{81}h^3 f^{///}(x_{2i}) + \frac{32}{243}h^4 f^{(4)}(\lambda_{4,2i})$$

$$t_{2i-2} \le \lambda_{4,2i} \le x_{2i}$$

$$f^{/}(t_{2i}) = f^{/}(x_{2i}) + \frac{2}{3}hf^{//}(x_{2i}) + \frac{2}{9}h^2 f^{///}(x_{2i}) + \frac{4}{81}h^3 f^{(4)}(\lambda_{5,2i})$$

$$x_{2i} \le \lambda_{5,2i} \le t_{2i}$$

$$f^{//}(t_{2i-2}) = f^{//}(x_{2i}) - \frac{4}{3}hf^{///}(x_{2i}) + \frac{8}{9}h^2 f^{(4)}(\lambda_{6,2i}), \quad t_{2i-2} \le \lambda_{6,2i} \le x_{2i}$$

$$f^{//}(t_{2i}) = f^{//}(x_{2i}) + \frac{2}{3}hf^{///}(x_{2i}) + \frac{2}{9}h^2 f^{(4)}(\lambda_{7,2i}), \quad x_{2i} \le \lambda_{7,2i} \le t_{2i}$$

$$f^{//}(x_{2i-2}) = f^{//}(x_{2i}) - 2hf^{///}(x_{2i}) + 2h^2 f^{(4)}(\lambda_{8,2i}), \quad x_{2i-2} \le \lambda_{8,2i} \le x_{2i}$$

$$f^{//}(x_{2i+2}) = f^{//}(x_{2i}) + 2hf^{///}(x_{2i}) + 2h^2 f^{(4)}(\lambda_{9,2i}), \quad x_{2i} \le \lambda_{9,2i} \le x_{2i+2}$$

$$f^{///}(t_{2i}) = f^{///}(x_{2i}) + \frac{2}{3}hf^{(4)}(\lambda_{10,2i}) \quad , \quad x_{2i} \le \lambda_{10,2i} \le t_{2i} \quad \Big] ....(5)$$

III. Proof of Theorem 1:

The proof depends on the following representation of $S_n(x)$ for $2ih \le x \le (2i+2)h$, $i = 0,1,....,m-1$ **we have**

$$S_n(x) = f(x_{2i})A_0(\frac{x-2ih}{2h}) + f(t_{2i})A_1(\frac{x-2ih}{2h}) + f(x_{2i+2})A_2(\frac{x-2ih}{2h})$$
$$+4h^2 S_n^{//}(x_{2i})A_3(\frac{x-2ih}{2h}) + 4h^2 S_n^{//}(x_{2i+2})A_4(\frac{x-2ih}{h}) \quad ....(6)$$

On using (3.0) and the conditions

$$S_n^{//}(0) = f^{//}(0) \ , \ S_n^{//}(1) = f^{//}(1) \qquad (7)$$

we say that $S_n(x)$ as given by (6) satisfies (1) is fourth in $[x_{2i}, x_{2i+2}], i = 0,1,2,....,m-1$.we also need to show that whether it is possible to determine $S^{//}{}_n(x_{2i})$ where $i = 1,2,....,m-1$ uniquely .For this purpose we use the fact that $S^{///}{}_n(x_{2i+2}) = S^{///}{}_n(x_{2i-}), i = 1,2,...,m-1$ with the help of (6) and (7) reduce to

$$\frac{41}{22}h^2 S^{//}{}_n(x_{2i-1}) + \frac{32}{22}h^2 S^{//}{}_n(x_{2i}) + \frac{35}{22}h^2 S^{//}{}_n(x_{2i+2}) =$$

$$\frac{243}{44}f(x_{2i}) - \frac{243}{44}f(t_{2i}) + \frac{81}{44}f(x_{2i+2}) - \frac{243}{44}f(t_{2i-2}) + \frac{81}{22}f(x_{2i-2}) \quad ....(8)$$

but (7) is a strictly tri diagonal dominant system has a unique solution[3] .

Thus $S^{//}{}_n(x_{2i}), i = 1,2,...,m-1$ can be obtained uniquely by the system (8)

This theorem is completed.

IV. ESITIMATES:

In order to prove theorem 1.2 , we need the following:

Lemma 4.0 :

Let $E_{2i} = \left| S^{//}{}_n(x_{2i}) - f^{//}(x_{2i}) \right|$ then for $f \in C^4[0,1]$ we have

$$\max E_{2i} \le \frac{81}{108}h^2 w(f^{(4)}; \frac{1}{m}) \ , \ i = 0,1,2,...,m-1 \qquad (9)$$

Proof:

From (8) we have

$$\left[\frac{41}{22}h^2(S^{//}{}_n(x_{2i-2}) - f^{//}(x_{2i-2})) + \frac{32}{22}h^2(S^{//}{}_n(x_{2i}) - f^{//}(x_{2i}))\right.$$

$$\left. + \frac{35}{22}h^2(S^{//}{}_n(x_{2i+2}) - f^{//}(x_{2i+2}))\right] = \left[\frac{243}{44}f(x_{2i}) - \frac{243}{44}f(t_{2i}) + \frac{81}{44}f(x_{2i+2}) - \right.$$

$$\left.\frac{243}{44}f(t_{2i-2}) + \frac{81}{22}f(x_{2i-2}) - \frac{41}{22}h^2 f^{//}(x_{2i-2}) - \frac{32}{22}h^2 f^{//}(x_{2i}) - \frac{35}{22}h^2 f^{//}(x_{2i+2})\right]$$

$$= \left[-\frac{2}{44}h^4 f^{(4)}(\lambda_{3,2i}) + \frac{54}{44}h^4 f^{(4)}(\lambda_{1,2i}) - \frac{32}{44}h^4 f^{(4)}(\lambda_{4,2i}) + \frac{54}{22}h^4 f^{(4)}(\lambda_{2,2i})\right.$$

$$\left. - \frac{82}{22}h^4 f^{(4)}(\lambda_{8,2i}) - \frac{70}{22}h^4 f^{(4)}(\lambda_{9,2i})\right] = \left[\frac{81}{22}\alpha_1 h^4 w(f^{(4)}, \frac{1}{m})\right], \ |\alpha_1| \le 1$$

The result (9) follows on using the property of diagonal dominant [3].

Lemma 4.1:

Let $f \in C^4[0,1]$ then

$(i)\ \left|S'''_n(x_{2i+}) - f'''(x_{2i})\right| \le \frac{135}{44} hw(f^{(4)}; \frac{1}{m})$

$(ii)\ \left|S'''_n(x_{2i-}) - f'''(x_{2i})\right| \le \frac{378}{88} hw(f^{(4)}; \frac{1}{m})$

$(iii)\ \left|S'''_n(t_{2i}) - f'''(t_{2i})\right| \le \frac{33}{22} hw(f^{(4)}; \frac{1}{m})$

$(iv)\ \left|S''_n(t_{2i}) - f''(t_{2i})\right| \le \frac{31}{44} h^2 w(f^{(4)}; \frac{1}{m})$

$(v)\ \left|S'_n(t_{2i}) - f'(t_{2i})\right| \le \frac{2546}{9801} h^3 w(f^{(4)}; \frac{1}{m})$

Proof:

From (6) we have

$$h^3 S'''_n(x_{2i+2}) = \frac{81}{22} f(x_{2i}) - \frac{243}{44} f(t_{2i}) + \frac{81}{44} f(x_{2i+2}) - \frac{123}{66} h^2 S''_n(x_{2i}) - \frac{39}{66} h^2 S''_n(x_{2i+2})$$

Hence

$$h^3 (S'''_n(x_{2i+}) - f'''(x_{2i})) = \frac{81}{22} f(x_{2i}) - \frac{243}{44} f(t_{2i}) + \frac{81}{44} f(x_{2i+2}) - \frac{123}{66} h^2 (S''_n(x_{2i}) - f''(x_{2i}))$$
$$- \frac{39}{66} h^2 (S''_n(x_{2i+2}) - f''(x_{2i+2})) - h^3 f'''(x_{2i}) - \frac{123}{66} h^2 f''(x_{2i}) - \frac{39}{66} h^2 f''(x_{2i+2})$$
$$= -\frac{1}{22} h^4 f^{(4)}(\lambda_{3,2i}) + \frac{27}{22} h^4 f^{(4)}(\lambda_{1,2i}) - \frac{39}{33} h^4 f^{(4)}(\lambda_{4,2i}) - \frac{123}{66} h^2 (S''_n(x_{2i}) - f''(x_{2i}))$$
$$- \frac{39}{66} h^2 (S''_n(x_{2i+2}) - f''(x_{2i+2}))$$

$$= \frac{27}{22} h^4 \alpha_2 w(f^{(4)}; \frac{1}{m}) - \frac{123}{66} (S''_n(x_{2i}) - f''(x_{2i})) - \frac{39}{66} h^2 (S''_n(x_{2i+2}) - f''(x_{2i+2}))\ ;\ |\alpha_2| \le 1$$

By using (9) the lemma (4.1)(i) follows ,the proof of lemma (4.1)(ii-v) are similar ,we have only mention that

$$h^3 S'''_n(x_{2i-}) = -\frac{81}{44} f(x_{2i}) + \frac{243}{44} f(t_{2i-2}) - \frac{81}{22} f(x_{2i-2}) + \frac{57}{66} h^2 S''_n(x_{2i-2}) + \frac{105}{66} h^2 S''_n(x_{2i})$$

$$h^3 S'''_n(t_{2i}) = \frac{27}{22} f(x_{2i}) + \frac{81}{44} f(t_{2i}) + \frac{27}{44} f(x_{2i+2}) - \frac{63}{66} h^2 S''_n(x_{2i}) + \frac{9}{66} h^2 S''_n(x_{2i+2})$$

$$h^2 S''_n(t_{2i}) = \frac{18}{11} f(x_{2i}) - \frac{27}{11} f(t_{2i}) + \frac{9}{11} f(x_{2i+2}) + \frac{2}{33} h^2 S''_n(x_{2i}) - \frac{5}{33} h^2 S''_n(x_{2i+2})$$

and

$$h S'_n(t_{2i}) = \frac{12}{11} f(x_{2i}) + \frac{39}{44} f(t_{2i}) - \frac{37}{44} f(x_{2i+2}) + \frac{34}{297} h^2 S''_n(x_{2i}) - \frac{14}{297} h^2 S''_n(x_{2i+2})$$

Proof of theorem (2):

For $0 \le Z \le 1$, we have

$$A_0(Z) + A_1(Z) + A_2(Z) = 1 \qquad ....(10)$$

let $x_{2i} \le Z \le x_{2i+2}$. On using (10) and (7) , we obtain

$$S'''_n(x) - f'''(x) = (S'''_n(x_{2i}) - f'''(x))A_0(\frac{x-2ih}{2h}) + (S'''_n(x_{2i+2}) - f'''(x))A_1(\frac{x-2ih}{2h})$$
$$+ (S'''_n(t_{2i}) - f'''(x))A_2(\frac{x-2ih}{2h}) = L_1 + L_2 + L_3 \qquad \ldots\ldots\ldots\ldots\ldots(11)$$

from (4) it follows that $|A_0(x)| \le 1,\ |A_1(x)| \le 1\ \ and\ \ |A_2(x)| \le 1$

Since $f'''(x) = f'''(x_{2i}) + (x - x_{2i}) f^{(4)}(\lambda)\ ,\ \ x_{2i} \le \lambda \le x$

Therefore

$L_1 = (S'''_n(x_{2i}) - f'''(x))A_0(\frac{x-2ih}{2h}) = (S'''_n(x_{2i}) - f'''(x_{2i}) - (x - x_{2i}) f^{(4)}(\lambda))A_0(\frac{x-2ih}{2h})$ On using lemma (4.1)(i) and $|x - x_{2i}| \le 2h,\ we\ \ obtain$

$$|L_1| \le |S'''_n(x_{2i}) - f'''(x_{2i})| + |(x - x_{2i})| \|f^{(4)}(\lambda)\| A_0(\frac{x-2ih}{2h})|$$

Therefore

$$|L_1| \le \frac{135}{44} h w(f^{(4)};\frac{1}{m}) + 2h\|f^{(4)}\| \qquad (12)$$

Similarly

$$|L_2| \le \frac{378}{88} h w(f^{(4)};\frac{1}{m}) + 2h\|f^{(4)}\|$$

$$L_3 = (S'''_n(t_{2i}) - f'''(x))A_2(\frac{x-2ih}{2h}) = (S'''_n(t_{2i}) - f'''(t_{2i}) + f'''(t_{2i}) - f'''(x_{2i})$$
$$- (x - x_{2i}) f^{(4)}(\lambda))A_2(\frac{x-2ih}{2h}) \qquad (13)$$

From (5) we have

$$f'''(t_{2i}) - f'''(x_{2i}) = \frac{2}{3} h f^{(4)}(\lambda_{10,2i})$$

where $x_{2i} \le \lambda_{10,\,2i} \le t_{2i}$, but $(3t_{2i} - x_{2i}) = 2h$

Hence $|L_3| \le |(S'''_n(t_{2i}) - f'''(t_{2i})| + 2h |f^{(4)}(\lambda_0) - f^{(4)}(\lambda)|,\quad x_{2i} \le \lambda, \lambda_0 \le x$

On using the above result with lemma (4.1)(iii), we obtain

$$|L_3| \le \frac{77}{22} h w(f^{(4)};\frac{1}{m}) \qquad \ldots\ldots(14)$$

putting (12) ,(14) in (11) ,we obtain

$$|S'''_n(x) - f'''(x)| \le \frac{956}{88} h w(f^{(4)},\frac{1}{m}) + 4h\|f^{(4)}\| \qquad \ldots\ldots.(15)$$

this prove (2) for r=3 to prove (2), when r=2

Since $S''_n(x) - f''(x) = \int_{t_{2i}}^{x} (S'''_n(t) - f'''(t))dt + (S''_n(t_{2i}) - f''(t_{2i}))$

on using lemma (4.1)(iv) and (15) we obtain

$$|S''_n(x) - f''(x)| \le \frac{987}{44} h^2 w(f^{(4)},\frac{1}{m}) + 8h^2\|f^{(4)}\| \qquad \ldots\ldots.(16)$$

which is proof (2) for r=2
the proof (2) for r=1
since

$$S'_n(x) - f'(x) = \int_{t_{2i}}^{x} (S''_n(t) - f''(t))dt + S'_n(t_{2i}) - f'(t_{2i})$$

On using lemma (4.1)(v) and (16),we get

$$\left|S^{l}{}_{n}(x)-f^{l}(x)\right|\leq\frac{884509}{19602}h^{3}w(f^{(4)},\frac{1}{m})+16h^{3}\left\|f^{(4)}\right\| \quad ....(17)$$

Which proof (2) for r=1, if r=0, we have

$$S_{n}(x)-f(x)=\int_{t_{2i}}^{x}(S^{l}{}_{n}(t)-f^{l}(t))dt+S_{n}(t_{2i})-f(t_{2i})$$

$$\sin ce$$

$$S_{n}(t_{2i})-f(t_{2i})=0$$

Thus

$$\left|S_{n}(x)-f(x)\right|\leq\frac{884509}{9801}h^{4}w(f^{(4)},\frac{1}{m})+32h^{4}\left\|f^{(4)}\right\|$$

Since 2mh=1 , then $h=\frac{1}{2m}$ put it in above result this completes the proof of theorem (1.2) .

V. CONCLUSIONS:

Throughout this paper we mentioned that some time the changes of boundary conditions effect on minimizing error bound in the subject of lacunary interpolation by Spline function .

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