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P-Points in the Construction of the Real Line

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Abstract— **Contemporary constructions of the real number line, via Dedekind cuts or equivalence classes of Cauchy sequences, make no use of infinitesimals. However, using another, lesser known approach, one shows easily that the field of real numbers, in whatever way it is constructed, is distilled from a larger ordered field that includes infinitesimals. In particular, given any complete, Archimedean, ordered field ℝ and any nonprinicpal ultrafilter *U* on the set ℕ of natural numbers, one may obtain the complete, non-Archimedean ordered field $F = Q^{\mathbb{N}}/U$ – where *Q* is the usual representative of the set of rational numbers in *F*. Then ℝ may be recovered, up to ordered field isomorphism, by forming the quotient F_{fin} / *I*, where F_{fin} is the set of finite elemnents of *F*, and *I* is the ideal of infinitesimals in *F*. We show in this paper that two natural criteria for deciding whether an element of $Q^{\mathbb{N}}/U$ is an infiinitesimal are equivalent if and only if the ultrafilter *U* is a *P-point*. A P-point is a nonprincipal ultrafilter *U* with the property that whenever $\{C_n : n \in \mathbb{N}\}$ is a partition of ℕ by sets not belonging to *U*, there is a $C \in U$ such that $C \cap C_n$ is finite for every $n \in \mathbb{N}$. It is known that the existence of P-points is independent of the usual (ZFC) axioms of set theory. This observation shows that undecidable objects and independence results play an essential role in mathematical constructions even as fundamental as the construction of the real line.**

Key words **— construction of the reals, infinitesimal, nonstandard analysis, P-points.**

I. Introduction

Many questions that one can ask about the set ℝ of reals cannot be decided by the usual axioms of set theory (ZFC). The most famous example of this phenomenon is the Continuum Hypothesis [7] which says that the cardinality of ℝ is undecidable. We show here that the very construction of ℝ from the field of rationals is tied to the existence of undecidable objects, namely, *P-points*. A P-point is a nonprincipal ultrafilter (defined below) U with the property that whenever $\{C_n : n \in \mathbb{N}\}$ is a partition of ℕ by sets not belonging to *U*, there is a $C \in U$ such that $C \cap C_n$ is finite for every $n \in \mathbb{N}$. It is well-known that whether P-points exist is undecidable from the standard (ZFC) axioms of set theory [7].

One of the lesser known ways of constructing ℝ from the field ℚ of rationals (e.g. [2,3,4]) involves forming the quotient of the set X_{fin} of "finite" elements of any complete, non-Archimedean ordered field X by the ideal *I* of infinitesimals. With this approach, it is easy to see that, no matter how ℝ is constructed initially, one may recover from ℝ a (fairly canonical) complete, non-Archimedean, ordered field by considering the ultrapower construction $X = \mathbb{Q}^{\mathbb{N}}/U$, where ℚ is the set of rationals defined in ℝ, and U is a nonprincipal ultrafilter on ℕ; then $\mathbb{R} \cong X_{fin}/I$.

In this note, we show that two natural criteria for deciding whether a member of $\mathbb{Q}^{\mathbb{N}}/U$ is an infinitesimal element are provably equivalent if and only if *U* itself is a P-point. By our observation in the previous paragraph, this shows that P-points play an essential role in the construction of ℝ, even though existence of P-points is undecidable.

Our work is organized as follows. We start with a short section on preliminaries, defining terms, establishing notation, and stating background results. In the second and final section, we prove the results mentioned in the previous paragraph.

II. Preliminaries

Suppose *F* is an ordered field . Then *F* has characteristic 0 and therefore contains copies of ℕ, ℤ, and ℚ (the sets of natural numbers, integers, and rationals, respectively). *F* is *Archimedean* if for every $x \in F$, there is $n \in \mathbb{N}$ with $x \le n$. ℚ is an Archimedean ordered field [6].

F is *Dedekind-complete* if every nonempty subset of *F* having an upper bound has a least upper bound. *F* is *complete* if every nested sequence $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ of closed intervals has a common point.

The next two theorems are standard (e.g. [1,8]).

Theorem 2.1 (*Completeness Equivalents*). *For an ordered field F, the following are equivalent.*
(1) *F is Dedekind-complete.*
(2) *F is Archimedean and complete.* ■

An *ordered field isomorphism* from an ordered field *K* to an ordered field *F* is a bijection $g: K \to F$ such that, for all $x, y \in K$,
(1) $g(x + y) = g(x) + g(y)$
(2) $g(x \cdot y) = g(x) \cdot g(y)$
(3) whenever $x < y$, both in *K*, $g(x) < g(y)$ in *F*.
When (1) – (3) hold, *K* and *F* are said to be *ordered field isomorphic*.

Theorem 2.2 (*Uniqueness*). *Any two complete, Archimedean, ordered fields are ordered field isomorphic.* ■

Suppose *F* is an ordered field. An *infinitesimal* in *F* is an $i \in F$ such that, for all $k \in \mathbb{N}, |i| < \frac{1}{k}$. An *infinite element* in *F* is a $j \in F$ such that, for all $n \in \mathbb{N}, |j| > n$. A *finite element* in *F* is any element that is not infinite. The next Proposition has a routine verification.

Proposition 2.3 (*Non-Archimedean Equivalents*). *Suppose F is an ordered field. The following are equivalent.*
(1) *F is non-Archimedean.*
(2) *F has an infinitesimal.*
(3) *F has an infinite element.* ■

A *nonprincipal ultrafilter on* ℕ is a set *U* of subsets of ℕ with the following properties:
(1) $\mathbb{N} \in U$

(2) for all $X, Y \in U, X \cap Y \in U$
(3) for all $X \in U$, if $X \subseteq Y \subseteq N$, then $Y \in U$
(4) for all $X \subseteq \mathbb{N}$, either $X \in U$ or $\mathbb{N} - X \in U$
(5) each $n \in \mathbb{N}, \{n\} \notin U$.

A consequence of the Axiom of Choice is that nonprincipal ultrafilters on $\mathbb{N}$ exist [7]:

Theorem 2.4 (*Ultrafilter Theorem*). *Nonprincipal ultrafilters on* $\mathbb{N}$ *exist.* ■

A *P-point* is a nonprincipal ultrafilter *U* with the following property: Whenever $\{C_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ by sets not belonging to U, there is a $C \in U$ such that $C \cap C_n$ is finite for every $n \in \mathbb{N}$.

The key result about P-points that will concern us in this note is the following.

Theorem 2.5 (*Independence of P-Points*). *The statement that P-points exist is independent of the axioms of set theory.* ■

The theorem follows from the following facts: (1) P-points can be constructed using the Continuum Hypothesis or Martin's Axiom, but (2) assuming ZFC is consistent, there is a model of set theory in which there are no P-points. See [7] for more on ultrafilters and P-points.

For any set *A*, we denote by $A^{\mathbb{N}}$ the set of all sequences whose terms lie in *A*. Let U be a nonprincipal ultrafilter on $\mathbb{N}$, and suppose $s, t \in \mathbb{Q}^{\mathbb{N}}$. We say *s is equivalent to t mod U* and write $s \sim t$ (or $s \sim_U t$ when necessary) if $\{n \in \mathbb{N} \mid s_n = t_n\} \in U$. Let $[s] = \{t \in \mathbb{Q}^{\mathbb{N}} \mid s \sim t\}$. We denote by $\mathbb{Q}^{\mathbb{N}}/U$ the set $\{[s] \mid s \in \mathbb{Q}^{\mathbb{N}}\}$. Define the *canonical map* $h: Q \to \mathbb{Q}^{\mathbb{N}}/U$ by $h(q) = [c^q]$, where c^q is the constant sequence $\langle q, q, q, \ . \ . \ . \ \rangle$. Define operations $+$ and $\cdot$ and an order relation $<$ on $\mathbb{Q}^{\mathbb{N}}$ pointwise. Then define operations $+$ and $\cdot$ and an order relation $<$ on $\mathbb{Q}^{\mathbb{N}}/U$ as follows: For all $[s], [t] \in \mathbb{Q}^{\mathbb{N}}/U$,

- $[s] + [t] = [s + t]$
- $[s] \cdot [t] = [s \cdot t]$
- $[s] < [t]$ if and only if $\{n \in \mathbb{N} \mid s_n < t_n\} \in U$.

These definitions, which do not depend on the choice of representatives, turn $\mathbb{Q}^{\mathbb{N}}/U$ into an ordered field. In fact, we have the following (e.g. [2,4,5]):

Theorem 2.6. $(\mathbb{Q}^{\mathbb{N}}/U, +, \ \cdot, <)$ *is a complete non-Archimedean ordered field.* ■

Letting $X = \mathbb{Q}^{\mathbb{N}}/U$, it is easy to verify that the set Q_X of rationals, defined in the standard way inside *X* as a field, is field isomorphic to the image of $\mathbb{Q}$ under the canonical map: $Q_X \cong h[Q] = \{[c^q] : q \in \mathbb{Q}\}$.

Suppose *X* is a complete non-Archimedean ordered field. Let *z* denote its zero element and *I* its set of infinitesimals. For $x, y \in X$, we write $x \simeq y$ if and only if $x - y \in I$. Let X_{fin} denote the set of finite elements of *X*. X_{fin} is an ordered ring and I is a maximal ideal in X_{fin}. Therefore, X_{fin}/I is a field; moreover, it becomes an ordered field by defining the less than relation < by:

$$x + I < y + I \text{ if and only if } x < y \text{ and } x \not\simeq y.$$

In fact, we have the following [2,4,5], where + and · denote the derived operations for the quotient (X_{fin} / I):

Theorem 2.7. $(X_{fin}/\mathrm{I}, +, \ \cdot, <)$ *is a complete Archimedean ordered field.* ■

The next theorem tells us that the real number line, no matter how it is constructed, "arises" from a complete ordered field with infinitesimals.

Theorem 2.8 (*Collapsing Theorem*). *Every complete Archimedean ordered field is obtained, up to ordered field isomorphism, as the quotient of the set of finite elements of a complete non-Archimedean ordered field by its set of infinitesimals.*

Proof. Let *F* be a complete, Archimedean ordered field, and let U be a nonprincipal ultrafilter on $\mathbb{N}$. Let $X = Q^{\mathbb{N}}/U$, where Q is the subfield of rationals defined in *F*. Then X_{fin}/I is a complete Archimedean ordered field, and is therefore ordered field isomorphic to *F*. ■

III. Main Results

For this section, given a nonprincipal ultrafilter *U* on $\mathbb{N}$, we let $X = X_U = \mathbb{Q}^{\mathbb{N}}/U$. We let X_{fin} denote the set of finite elements of *X*.

Let *I* denote the set of infinitesimals in X_{fin}. By definition, *I* is the set $\{[s] \in X_{fin} : \forall\, k \in \mathbb{N}\ (|[s]| < [c^{1/k}]\}$ (where $|[s]|$ signifies the *absolute value* of $[s]$, computed in X_{fin}). Therefore, *I* can be defined by

$$I = \{[s] \in X_{fin} : s \in S\},$$

where, for all $s \in \mathbb{Q}^{\mathbb{N}}/U$,

$$s \in S \iff \forall\, k \in \mathbb{N}\ (\{n \in \mathbb{N} : |s_n| < \frac{1}{k}\} \in U)$$

Note that, as usual, this formulation does not depend on the choice of representatives for $[s]$.

Considering that, whenever $s = \langle s_1, s_2, s_3, \ldots \rangle$ is a sequence of rationals that converges to 0, $[s] \in I$ (and hence, $[s] \in S$), the following notion is natural: We will say that a sequence s of rationals *converges mod U to 0* and write $s \xrightarrow{U} 0$, if there is a set $B \in U$ such that for each $k \in \mathbb{N}$, there is $n \in \mathbb{N}$ so that $|s_m| < \frac{1}{k}$ for all $m \in B$ for which $m \geq n$. This notion leads to analogues of *S* and *I* as follows: For each nonprincipal ultrafilter *U* on $\mathbb{N}$, we define:

$$S_U = \{s \in \mathbb{Q}^{\mathbb{N}} : s \xrightarrow{U} 0\}$$

$$I_U = \{[s] \in X_{fin} : s \in S_U\}.$$

Again, the criterion of membership for $[s]$ in I_U does not depend on the choice of representatives.

We ask now whether the different notions of "infinitesimal" suggested by the defintions of I and I_U are equivalent. The next easy lemma shows that the I_U-version is at least as strong as the *I*-version.

Lemma 3.1. *For evey nonprincipal ultrafilter* U *on* $\mathbb{N}$, $S_U \subseteq S$ *and* $I_U \subseteq I$. ■

Proof. Suppose $s \in S_U$ and let $B \in U$ be a witness. Let $k \in \mathbb{N}$. Let n be such that for all $m \in B$ for which $m \geq n$,

$|s_m| < \frac{1}{k}$. Since $B_n = \{m \in B : m \geq n\} \in U$ and $B_n \subseteq S(k) = \left\{m : |s_m| < \frac{1}{k}\right\}$, it follows that $S(k) \in U$. We have shown $s \in S$. Hence $S_U \subseteq S$. It follows that $I_U \subseteq I$. ■

We now show that whether $I_U \supseteq I$ is independent of ZFC, because the truth of the statement depends on whether U is a P-point.

Theorem 3.2. *Let U be a nonprincipal ultrafilter on $\mathbb{N}$. The following statements are equivalent (in ZFC):*
(1) $I_U \supseteq I$
(2) *U is a P-point.*

Note here that $I_U \supseteq I$ if and only if $S_U \supseteq S$.

Proof of $(1) \Longrightarrow (2)$. Let $\{A_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ so that $A_n \notin U$ for each n. Define $B_1, B_2, B_3, \ldots, B_n, \ldots$ by
$$B_n = \mathbb{N} - (A_1 \cup A_2 \cup \cdots \cup A_n).$$
Clearly, $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$, and for each $n \in \mathbb{N}$, $B_n \in U$. For each $m \in \mathbb{N}$, define the mth term of a sequence s of rationals by
$$s_m = \frac{1}{n} \Leftrightarrow m \in A_n.$$

Claim 1. For each $k \in \mathbb{N}$,
$$B_k = \left\{m \in \mathbb{N} : s_m < \frac{1}{k}\right\}.$$

Proof of Claim 1. Proceed by induction on k. For $k = 1$, this follows from the definition of A_1 and the fact that $s_m \leq 1$ for every $m \in \mathbb{N}$.

Assuming the result for k, we show $B_{k+1} = \left\{m \in \mathbb{N} : s_m < \frac{1}{k+1}\right\}$. Using the definitions of A_k and B_k, we have:
$$\begin{aligned} B_{k+1} &= \mathbb{N} - (A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}) \\ &= \mathbb{N} - (A_1 \cup A_2 \cup \ldots \cup A_k) - A_{k+1} \\ &= \left\{m \in \mathbb{N} : s_m < \frac{1}{k}\right\} - \left\{m \in \mathbb{N} : s_m = \frac{1}{k+1}\right\} \\ &= \left\{m \in \mathbb{N} : s_m < \frac{1}{k+1}\right\}. \end{aligned}$$
This completes the induction and the proof of Claim 1. ■

Claim 2. $[s] \in I$ and $s \in S$.

Proof of Claim 2. We show $[s]$ is an infinitesimal by showing $[s] < [c^{1/k}]$ for every $k \in \mathbb{N}$ (recall that s is positive, by definition). But this follows from Claim 1, since $B_k = \{m \in \mathbb{N} : s_m < 1/k\}$ and $B_k \in U$. ■

Claim 3. U is a P-point.

Proof of Claim 3. We obtain $B \in U$ such that $B \cap A_n$ is finite for each $n \in \mathbb{N}$. By hypothesis and Claim 2, $s \in S_U$. Let $B \in U$ witness the fact that $s \in S_U$. Then for each $k \in \mathbb{N}$, there is n_k such that for all $m \in B$ with $m \geq n_k$, we have $s_m < 1/k$.

Suppose $B \cap A_k$ is infinite for some k. Then, by definition of A_k, it follows that there are infinitely many m for which $s_m \geq 1/k$, and this contradicts the existence of a value n_k, described in the previous paragraph. Therefore, $B \cap A_k$ is finite for all $k \in \mathbb{N}$. We have shown that U is a P-point. ■

Proof of $(2) \Longrightarrow (1)$. Suppose U is a P-point. We show that $I_U \supseteq I$. Let $[s] \in I$. If $s \simeq z$, $[s] \in I_U$, and we are done, so assume $s \not\simeq z$. (Recall that z is the zero sequence in $\mathbb{Q}^{\mathbb{N}}$.)

For each $k \in \mathbb{N}$, we have
$$[c^0] < |[s]| < [c^{1/k}]$$
so the set $B_k = \left\{m \in \mathbb{N} : 0 < |s_m| < \frac{1}{k}\right\}$ belongs to U. Clearly, $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$, and $\bigcap_n B_n = \emptyset$.

Define $A_1, A_2, A_3, \ldots$ as follows: $A_1 = \mathbb{N} - B_1$, and for each $n > 1$, $A_{n+1} = B_n - B_{n+1}$. The A_n form a partition of $\mathbb{N}$. Since U is a P-point, we can find $B \in U$ such that $B \cap A_n$ is finite for every n. We show s tends to 0 on B. For each $i \in \mathbb{N}$, if $B \cap A_i = \emptyset$, let $m_i = 1$; otherwise, define m_i to be $\max(B \cap A_i)$. Let $k \in \mathbb{N}$. Pick $n \geq \max(\{m_i : 1 \leq i \leq k\})$. If $m \geq n$ and $m \in B$, then $m \notin A_1 \cup A_2 \cup \cdots \cup A_k$. We will be able to conclude that $m \in B_k$, whence $|s_m| < \frac{1}{k}$, once the following final claim (Claim 4) has been proven; since k is arbitrary, this will complete the proof of $(2) \Longrightarrow (1)$. ■

Claim 4. For each $k \in \mathbb{N}$, $B - (A_1 \cup A_2 \cup \cdots \cup A_k) \subseteq B_k$.

Proof of Claim 4. Proceed by induction on k. For $k = 1$:
$$\begin{aligned} B - A_1 &= B - (\mathbb{N} - B_1) \\ &= B \cap B_1 \\ &\subseteq B_1. \end{aligned}$$
Assuming the result for k, we have
$$\begin{aligned} &B - (A_1 \cup A_2 \cup \cdots \cup A_{k+1}) \\ &= B - (A_1 \cup A_2 \cup \cdots \cup A_k) - A_{k+1} \\ &\subseteq B_k - A_{k+1} \\ &= B_{k+1}. \end{aligned}$$ ■

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