Characterizations of Vague Groups

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Abstract— In this present paper the author describes application of vague groups, although the concept of vague groups was defined by Rosenfeld [8] is the first application of fuzzy theory in Algebra. Since then a number of works have been done in the area of fuzzy algebra. In this paper we present the notion of Intuitionistic fuzzy groups and make some characterizations of them.

Index Terms—Vague, Algebra, Fuzzy group.

I. INTRODUCTION
The theory of vague groups play an important role in the field of soft computing, automata, decision sciences, mathematical logics, and many more. Since there are a number of generalizations of Zadeh’s fuzzy set theory and fuzzy algebra, two-fold fuzzy theory, intuitionistic fuzzy theory, L-fuzzy theory, etc. ([1], [2], [3], [4], [5], [6], [7]) While fuzzy sets are applicable to each of such application domains, higher order fuzzy sets cannot, because of their specialization, in particular, this is a higher order of fuzzy sets makes the solution procedure more complex, but if the complexity of computation time, computation volume memory-space is not the matter of concern then a better results could be achieved. Since the inception of the theory of fuzzy algebra by Rosenfeld, a good amount of work have been reported so far by various authors in this area. The vague groups have an extra edge over fuzzy groups. Consequently there is a reason that intuitionistic fuzzy algebra will play a significant role in the area of fuzzy algebra in due time.

II. INTUITIONISTIC FUZZY GROUPS (OR VAGUE GROUPS)
Some Results
The theory of fuzzy groups defined by Rosenfeld [100] is the first application of fuzzy theory in Algebra. Since then a number of works have been done in the area of fuzzy algebra. In this section we present the notion of intuitionistic fuzzy (vague) groups and make some characterizations of them. First of all we recollect the following notations on interval arithmetic.

Some Notations
Let I[0,1] denotes the family of all closed subintervals of [0,1]. If I₁ = [a₁, b₁] and I₂ = [a₂, b₂] be two elements of I[0,1], we call I₁ ⊇ I₂ if a₁ ≥ a₂ and b₁ ≥ b₂. Similarly we understand the relations I₁ ⊆ I₂ and I₁ = I₂. Clearly the relation I₁ ⊇ I₂ does not necessarily imply that I₁ ⊆ I₂ and conversely. Also for any two unequal intervals I₁ and I₂, there is no necessity that either I₁ ≥ I₂ or I₁ ≤ I₂ will be true.

The term ‘imax’ means the maximum of two intervals as $\text{imax}(I₁, I₂) = [\max(a₁, a₂), \max(b₁, b₂)]$. Similarly defined is ‘imin’. The concept of ‘imax’ and ‘imin’ could be extended to define ‘isup’ and ‘iinf’ of infinite number of elements of I[0,1]. It is obvious that $L = \{[0,1], \text{isup}, \text{iinf}, \leq, \} \text{ is a lattice with universal bounds } [0,0] \text{ and } [1,1]$. Now we give the definition of intuitionistic fuzzy groups.

Definition 1.2
Let (X, *) be a group. An intuitionistic fuzzy set A of X is called an intuitionistic fuzzy group (IFG) of X if the following conditions are true:

\[ \forall x, y \in X, \quad V_A(xy) \geq \min \{V_A(x), V_A(y)\} \quad \text{and} \quad V_A(x^1) \geq V_A(x). \]

(i) $t_A(xy) \geq \min \{t_A(x), t_A(y)\}$, $1 - f_A(xy) \geq \min \{1 - f_A(x), 1 - f_A(y)\}$ and

(ii) $t_A(x^1) \geq t_A(x)$

and

$1 - f_A(x^1) \geq 1 - f_A(x).$ (Here the element xy stands for x*y).

Example
Consider the group X = {1, ω, ω²} with respect to the binary operation ‘complex number multiplication’, where ω is the imaginary cube root of unity. Clearly, the intuitionistic fuzzy set A = {(1, .9, .1), (ω, .6,.2), (ω²,.4,.6)} is an intuitionistic fuzzy group (IFG) of the group X.

Using our short notation, we reproduce the following definitions earlier given by Atanassov. Let A and B be two IFSs of the universe U. Then

(i) $A = B$ if $V_A(u) = V_B(u)$

(ii) $A \subset B$ if $V_A(u) \leq V_B(u)$

(iii) $C = A \cup B$ if $V_C(u) = \text{imax} \{V_A(u), V_B(u)\}$

(iv) $C = A \cap B$ if $V_C(u) = \text{iinf} \{V_A(u), V_B(u)\}$

The complement of an IFS of A of the universe U is the IFS $A^c$ given by the IFS values $V_A^c(u) = 1 - V_A(u)$.

The following propositions are straightforward.

Proposition 1.3
If A is an intuitionistic fuzzy group of a group X, then $\forall x \in X, \quad V_A(x^1) = V_A(x)$ i.e. $t_A(x^1) \geq t_A(x)$ and $1 - f_A(x^1) = 1 - f_A(x)$.

Proposition 1.4
Zero intuitionistic fuzzy set, unit intuitionistic fuzzy set and all α-IF sets of a group X are trivial IFGs of X.

Proposition 1.5
A necessary and sufficient condition for an intuitionistic fuzzy set of a group X to be a IFG group of X is that $V_A(xy^1) \geq \min \{V_A(x), V_A(y)\}$. Proof: Let A be an IFG of the group X. Then $t_A(xy^1) \geq \min \{t_A(x), t_A(y^1)\} \geq \min \{t_A(x), t_A(y)\}$. Similarly, $1 - f_A(xy^1) \geq \min \{1 - f_A(x), 1 - f_A(y)\}$. For the converse part, suppose that A be an IFG of a group X of which e is the identity element. Now $t_A(e) \geq \min \{t_A(x), t_A(y)\}$

Now $t_A(x) \geq \min \{t_A(x), t_A(y)\}$

or $t_A(e) \geq t_A(y)$. Now $t_A(e) \geq \min \{t_A(x), t_A(y)\}$

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or, \( t_\alpha(y^{-1}) \geq t_\alpha(y) \), which gives \( v_\alpha(y^{-1}) = t_\alpha(y) \).

Also \( v_\alpha(xy) \geq \min \{ t_\alpha(x), t_\alpha(y) \} \).

Similarly it can be proved that \( 1 - f_\alpha(x') \geq 1 - f_\alpha(x) \), and \( 1 - f_\alpha(xy) \geq \min \{ 1 - f_\alpha(x), 1 - f_\alpha(y) \} \).

**Proposition 1.5**

If A and B are two IFGs of a group X, then \( A \cap B \) is also an IFG of X.

Proof: Let \( A = (x, t_\alpha, t_\beta) \) be an IFG of a group X. Then for \( \alpha, \beta \in [0,1] \), the \( \alpha \)-cut and \( \beta \)-cut of A are defined below in the following way.

\[ A_{\alpha,\beta} = \{ x : x \in X, \ V_\alpha(x) \geq [\alpha, \beta] \} \]

Clearly \( A_{0,0} = X \). The \( (\alpha,\beta) \)-cuts are also called IF-cuts of the IFS A.

**Definition 1.9**

The \( \alpha \)-cut of the IFS A is a crisp subset \( A_\alpha \) of the set X given by

\[ A_\alpha = A_{\alpha,\alpha} \]

Thus \( A_0 = X \) and if \( \alpha \geq \beta \) then \( A_\beta \subseteq A_\alpha \) and \( A_{\alpha,\beta} = A_\alpha \).

Equivalently, we can define the \( \alpha \)-cut as

\[ A_\alpha = \{ x : x \in X, \ t_\alpha(x) \geq \alpha \} \.

**Proposition 1.10**

Let A be an IFG of a group X. Then for \( \alpha \in [0,1] \), the \( \alpha \)-cut \( A_\alpha \) is a crisp subgroup of X.

Proof: \( \forall x, y \in A_\alpha \) we have \( t_\alpha(x) \geq \alpha, t_\alpha(y) \geq \alpha \). Now \( t_\alpha(xy^{-1}) \geq \min \{ t_\alpha(x), t_\alpha(y) \} = \alpha \). Therefore, \( xy^{-1} \in A_\alpha \). Hence proved.

**Proposition 1.11**

Let A be an IFG of a group X. Then \( \forall \alpha, \beta \in [0,1] \), the \( (\alpha,\beta) \)-cut \( A_{\alpha,\beta} \) is a crisp subgroup of X.

Proof: \( \forall x, y \in A_{\alpha,\beta} \) we have \( t_\alpha(x) \geq \alpha, t_\beta(x) \geq \beta \). If \( t_\alpha(xy^{-1}) \geq \min \{ t_\alpha(x), t_\beta(y) \} = \alpha \).

Similarly, we see that \( 1 - f_\alpha(xy^{-1}) \geq \beta \). Therefore, \( xy^{-1} \in A_{\alpha,\beta} \). Hence proved.

The subgroups like \( A_{\alpha,\beta} \) are also called IF-cut subgroups of X.

**Proposition 1.12**

Let X be a group and A be an IFG of X. Two IF-cut subgroups \( A_{\alpha,\beta} \) and \( A_{\gamma,\omega} \) with \( [\alpha,\beta] < [\gamma,\omega] \) are equal iff there is no \( x \in X \) such that \( [\alpha,\beta] \leq V_\alpha(x) \leq [\gamma,\omega] \).

**Proposition 1.13**

Let X be a finite group of order n, and A be an IFG of X. Consider the set \( V(A) \) given by \( V(A) = \{ V_\alpha(x) : x \in X \} \). Then \( A_i \) are the only IF-cut groups of X, where \( i \in V(A) \).

Proof: Consider \( [x, y] \in [0,1] \) where \( [x, y] \notin V(A) \). If \( [x, y] < [x_i, y_i] \) where \( [x_i, y_i] \in V(A) \), then \( A_{x_i,y_i} = A_{1,i,2} = A_{0,0} \). If \( [x_i, y_i] < [x_i, y_i] \) where \( [x_i, y_i] \in V(A) \), then \( A_{x_i,y_i} = A_{1,i,3} = X = A_{1,1,2} \). Thus for any \( [x, y] \in [0,1] \), the IF-cut group \( A_{1,1,2} \) is one of \( A_i \) for \( i \in V(A) \). Hence proved.

**Proposition 1.14**

Any subgroup H of a group X is an IF-cut group of some IFG of X.

Proof: Consider the IFS A of X given by \( V_\alpha(X) = [t, t] \) if \( x \in H \) and \( [0,0] \) if \( x \notin H \). where \( t \in (0, 1) \).

It can be proved that \( \forall x, y \in X \),

\( V_\alpha(xy^{-1}) \geq \min \{ V_\alpha(x), V_\alpha(y) \} \).

\( \implies A \) is an IFG of X. Clearly \( H = A_{\alpha/\beta} \). Hence proved.

**Proposition 1.15**

Let X be a finite group and A be an IFG of X. Consider the subset H of X given by

\( \forall x, y \in H \), we have \( V_\alpha(xy^{-1}) \geq \min \{ V_\alpha(x), V_\alpha(y) \} \) if \( x \in H \).

Thussxy \in H. Hence proved.

**References**


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