Global attractivity and positive almost periodic solution of a discrete multispecies Gilpin-Ayala competition system with feedback control

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Abstract— **This paper discusses a discrete multispecies Gilpin-Ayala competition system. We first achieve the permanence of the system. Assume that the coefficients in the system are almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive. One example together with numerical simulation indicates the feasibility of the main results.**

Index Terms—**Permanence, Almost periodic solution, Discrete, Gilpin-Ayala competition system, Global attractivity**

I. INTRODUCTION

As we all known, investigating the almost periodic solutions of discrete population dynamics model with feedback control has more extensively practical application value(see [1–9] and the references cited therein).Wang [2] considered a nonlinear single species discrete with feedback control and obtain some sufficient conditions which assure the unique existence and global attractivity of almost positive periodic solution. Niu and Chen [5] studied a discrete Lotka-Volterra competitive system with feedback control and obtain the existence and uniqueness of the almost periodic solution which is uniformly asymptotically stable.

In this paper, we investigate the dynamic behavior of the following discrete multispecies Gilpin-Ayala competition model with feedback controls

$$\begin{cases} x_i(k+1) = x_i(k)\exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(x_j(k))^{\theta_{ij}} - e_i(k)u_i(k)\right\}, \\ \Delta u_i(k) = -f_i(k)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)x_j(k), \quad i = 1, 2, \cdots, n, \end{cases} \tag{1.1}$$

where $\{a_{ij}(k)\}$, $\{b_i(k)\}$, $\{e_i(k)\}$, $\{f_i(k)\}$ and $\{g_{ij}(k)\}$ are bounded nonnegative almost periodic sequences such that

$$\begin{aligned} &0 < a_{ij}^l \le a_{ij}(k) \le a_{ij}^u, \quad 0 < b_i^l \le b_i(k) \le b_i^u, \quad 0 < e_i^l \le e_i(k) \le e_i^u, \\ &0 < f_i^l \le f_i(k) \le f_i^u < 1, \quad 0 < g_{ij}^l \le g_{ij}(k) \le g_{ij}^u, \end{aligned} \tag{1.2}$$

$i, j = 1, 2, \cdots, n$, $k \in Z$.

Denote as Z and Z^+ the set of integers and the set of nonnegative integers, respectively. For any bounded sequence $\{g(n)\}$ defined on Z, define

$$g^u = \sup_{n\in Z} g(n), g^l = \inf_{n\in Z} g(n).$$

From the point of view of biology, in the sequel, we assume that $x(0) = (x_1(0), x_2(0), \cdots, x_n(0), u_1(0), u_2(0), \cdots, u_n(0)) > 0$. Then it is easy to see that, for given $x(0) > 0$, the system (1.1) has a positive sequence solution $x(k) = (x_1(k), x_2(k), \cdots, x_n(k))(k \in Z^+)$ passing through x(0).

With the stimulation from the works [10–13], the main purpose of this paper is to obtain a set of sufficient conditions to ensure the existence of a unique globally attractive positive almost periodic solution of system (1.1) with initial condition (1.3).

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In the next section, we establish the permanence of system (1.1). Then, in Section 4, we establish sufficient conditions to ensure the existence of a unique positive almost periodic solution which is globally attractive. The main results are illustrated by an example with numerical simulation in Section 5. Finally, the conclusion ends with brief remarks in the last section.

II. PRELIMINARIES

First, we give the definitions of the terminologies involved.

Definition 2.1([14]) A sequence x: Z→R is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : | x(n+\tau) - x(n) | < \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$| x(n+\tau) - x(n) | < \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of x(n).

Denition 2.2([15]) A sequence x : $Z^+ \to R$ is called an asymptotically almost periodic sequence if

$$x(n) = p(n) + q(n), \forall n \in Z^+,$$

where $p(n)$ is an almost periodic sequence and $q(n)\to 0$, $n\to\infty$.

Denition 2.3([16]) A solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) is said to be globally attractive if for any other solution $(x^*_1(k), x^*_2(k), \cdots, x^*_n(k))$ of system (1.1), we have

$$\lim_{k\to+\infty}(x_i^*(k) - x_i(k)) = 0, \ i = 1, 2, \cdots, n.$$

Now, we present some results which will play an important role in the proof of the main result.

Lemma 2.1([17]) If $\{x(n)\}$ is an almost periodic sequence, then $\{x(n)\}$ is bounded.

Lemma 2.2([18]) $\{x(n)\}$ is an almost periodic sequence if and only if, for any sequence $m_i \subset Z$, there exists a subsequence $\{m_{ik}\}\subset\{m_i\}$ such that the sequence $\{x(n+m_{ik})\}$ converges uniformly for all $n\in Z$ as $k\to\infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.3([15]) $\{x(n)\}$ is an asymptotically almost periodic sequence if and only if, for any sequence $m_i \subset Z$ satisfying $m_i > 0$ and $m_i\to\infty$ as $i\to\infty$ there exists a subsequence $\{m_{ik}\} \subset \{m_i\}$ such that the sequence $\{x(n + m_{ik})\}$ converges uniformly for all $n \in Z^+$ as $k\to\infty$.

Lemma 2.4([17]) Suppose that $\{p_1(n)\}$ and $\{p_2(n)\}$ are almost periodic real sequences. Then $\{p_1(n)+p_2(n)\}$ and $\{p_1(n)p_2(n)\}$ are almost periodic; $1/p_1(n)$ is also almost periodic provided that $p_1(n) \neq 0$ for all $n \in Z$.

Moreover, if $\varepsilon > 0$ is an arbitrary real number, then there exists a relatively dense set that is ε−almost periodic common to $\{p_1(n)\}$ and $\{p_2(n)\}$.

Lemma 2.5([19]) Assume that sequence $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x^{\alpha}(n)\} \tag{2.1}$$

for $n \in N$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n) \le \left(\frac{1}{\alpha b^l}\right)^{\frac{1}{\alpha}} \exp\{a^u - \frac{1}{\alpha}\}. \tag{2.2}$$

Lemma 2.6([20]) Assume that sequence $\{x(n)\}$ satisfies

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x^{\alpha}(n)\}, \quad n \ge N_0,$$

$$\limsup_{n\to+\infty} \le x^*$$

and $x(N_0)>0$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{n\to+\infty} x(n) \ge \min\left\{\left(\frac{a^l}{b^u}\right)^{\frac{1}{\alpha}} \exp\{a^l - b^u(x^*)^{\alpha}\}, \left(\frac{a^l}{b^u}\right)^{\frac{1}{\alpha}}\right\}. \tag{2.3}$$

Lemma 2.7([21]) Assume that $A > 0$ and $y(0) > 1$, and further suppose that

$$y(n+1) \le Ay(n) + B(n), \quad n = 1, 2, 3, \cdots.$$

Then for any integer $k \le n$,

$$y(n) \le A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1).$$

Especially, if $A < 1$ and B is bounded above with respect to M, then

$$\limsup_{n\to\infty} y(n) \le \frac{M}{1-A}.$$

Lemma 2.8([21]) Assume that $A > 0$ and $y(0) > 1$, and further suppose that

$$y(n+1) \ge Ay(n) + B(n), \quad n = 1, 2, 3, \cdots.$$

Then for any integer $k \le n$,

$$y(n) \ge A^k y(n-k) + \sum_{i=0}^{k-1} A^i B(n-i-1).$$

Especially, if $A < 1$ and B is bounded below with respect to m, then

$$\liminf_{n\to\infty} y(n) \ge \frac{m}{1-A}.$$

III. PERMANENCE

In this section, we establish the permanence result for system (1.1).

Proposition 3.1 Assume that the conditions (1.2) and (1.3) hold, furthermore,

$$b_i^l - e_i^u N_i > 0, \tag{3.1}$$

then system (1.1) is permanent, that is, there exist positive constants m_i, M_i, n_i and $N_i (i = 1, 2, \cdots, n)$ which are independent of the solutions of system (1.1), such that for any positive solution $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ of system (1.1), one has:

$$m_i \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_i,$$
$$n_i \le \liminf_{k\to+\infty} u_i(k) \le \limsup_{k\to+\infty} u_i(k) \le N_i, \qquad i = 1, 2, \cdots, n.$$

Proof. Let $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ be any positive solution of system (1.1). From the first equation of system (1.1), it follows that

$$x_i(k+1) \le x_i(k)\exp\left\{b_i(k) - a_{ii}(k)(x_i(k))^{\theta_{ii}}\right\}. \tag{3.2}$$

Thus, as a direct corollary of Lemma 2.5, according to (3.2), one has

$$\limsup_{k\to+\infty} x_i(k) \le \left(\frac{1}{\theta_{ii}a_{ii}^l}\right)^{\frac{1}{\theta_{ii}}} \exp\left\{b_i^u - \frac{1}{\theta_{ii}}\right\} \triangleq M_i > 0. \tag{3.3}$$

For any small positive constant $\varepsilon > 0$, from (3.3) it follows that there exists a positive constants $K_1 > 0$ such that for all $k > K_1$ and $i = 1, 2, \cdots, n$,

$$x_i(k) \le M_i + \varepsilon. \tag{3.4}$$

For $k \ge K_1$, from (3.4) and system (1.1), we have

$$u_i(k+1) \le (1 - f_i^l)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)(M_j + \varepsilon). \tag{3.5}$$

Then, as a direct corollary of Lemma 2.7, according to (3.5), one has

$$\limsup_{k\to+\infty} u_i(k) \le \frac{1}{f_i^l}\sum_{j=1}^{n} g_{ij}^u (M_j + \varepsilon).$$

Letting $\varepsilon \to 0$, it follows that

$$\limsup_{k\to+\infty} u_i(k) \le \frac{1}{f_i^l}\sum_{j=1}^{n} g_{ij}^u M_j \triangleq N_i. \tag{3.6}$$

Thus, there exists a positive integer $K_2 > K_1$, we have for $k > K_2$

$$u_i(k) \le N_i + \varepsilon. \tag{3.7}$$

For $k \ge K_2$, from (3.7) and system (1.1), we have

$$x_i(k+1) \ge x_i(k)\exp\left\{b_i(k) - a_{ii}(k)(x_i(k))^{\theta_{ii}} - e_i(k)(N_i + \varepsilon)\right\}. \tag{3.8}$$

Assuming that $b_i^l - e_i^u N_i > 0$, for any $\varepsilon > 0$, there exists a positive integer $K_3 > K_2$ such that $b_i(k) - e_i(k)(N_i + \varepsilon) > 0$ for $k > K_3$. Thus, as a direct corollary of Lemma 2.6, according to (3.8), one has

$$\liminf_{k\to+\infty} x_i(k) \ge \min\{m_{i_1\varepsilon}, m_{i_2\varepsilon}\},$$

where

$$m_{i_1\varepsilon} = \left(\frac{b_i^l - e_i^u(N_i + \varepsilon)}{a_{ii}^u}\right)^{\frac{1}{\theta_{ii}}},$$
$$m_{i_2\varepsilon} = m_{i_1\varepsilon}\exp\{b_i^l - e_i^u(N_i + \varepsilon) - a_{ii}^u(M_i)^{\theta_{ii}}\}.$$

Letting $\varepsilon \to 0$, it follows that

$$\liminf_{k\to+\infty} x_i(k) \ge \frac{1}{2}\min\{m_{i_1}, m_{i_2}\} \triangleq m_i > 0, \tag{3.9}$$

where

$$m_{i_1} = \left(\frac{b_i^l - e_i^u N_i}{a_{ii}^u}\right)^{\frac{1}{\theta_{ii}}},$$
$$m_{i_2} = m_{i_1}\exp\{b_i^l - e_i^u N_i - a_{ii}^u(M_i)^{\theta_{ii}}\}.$$

From (3.9), for any $\varepsilon > 0$, there exists a positive integer $K_4 > K_3$ such that

$$x_i(k) \ge m_i - \varepsilon \tag{3.10}$$

for $k > K_4$.

From (3.10) and system (1.1), we have

$$u_i(k+1) \ge (1 - f_i^u)u_i(k) + \sum_{j=1}^{n} g_{ij}(k)(m_j - \varepsilon). \tag{3.11}$$

Then, as a direct corollary of Lemma 2.8, according to (3.11), one has

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{f_i^u}\sum_{j=1}^{n} g_{ij}^l (m_j-\varepsilon).$$

Letting ε→0, it follows that

$$\liminf_{k\to+\infty} u_i(k) \geq \frac{1}{f_i^u}\sum_{j=1}^{n} g_{ij}^l m_j \triangleq n_i > 0. \tag{3.12}$$

Then, (3.3), (3.6), (3.9) and (3.12) show that system (1.1) is permanent. The proof is completed.

Theorem 3.2 Assume that (1.2), (1.3) and (3.1) hold, then system (1.1) is permanent.

We denote by Ω the set of all solutions $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ of system (1.1) satisfying $m_i \leq x_i(k) \leq M_i$, $n_i \leq u_i(k) \leq N_i (i = 1, 2, \cdots, n)$ for all $k \in Z^+$.

Proposition 3.3 Assume that (1.2), (1.3) and (3.1) hold. Then $\Omega \neq \Phi$.

Proof. By the almost periodicity of $\{a_{ij}(k)\}$, $\{b_i(k)\}$, $\{e_i(k)\}$, $\{f_i(k)\}$ and $\{g_{ij}(k)\}$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to +\infty$ as $p \to +\infty$ such that

$$a_{ij}(k+\delta_p)\to a_{ij}(k),\quad b_i(k+\delta_p)\to b_i(k),$$
$$g_{ij}(k+\delta_p)\to g_{ij}(k),\quad e_i(k+\delta_p)\to e_i(k),\quad f_i(k+\delta_p)\to f_i(k),$$
$$\text{as}\quad p\to+\infty.$$

Let ε be an arbitrary small positive number. It follows from Theorem 3.3 that there exists a positive integer N_0 such that

$$m_i-\varepsilon \leq x_i(k) \leq M_i+\varepsilon,\quad n_i-\varepsilon\leq u_i(k)\leq N_i+\varepsilon,\quad k>N_0.$$

Write $x_{ip}(k) = x_i(k + \delta_p)$ and $u_{ip}(k) = u_i(k + \delta_p)$ for $k \geq N_0 - \delta_p$ and $p = 1, 2, \cdots$. For any positive integer q, it is easy to see that there exists a sequence $\{x_{ip}(k) : p \geq q\}$ such that the sequence $x_p(k)$ has a subsequence, denoted by $\{x_{ip}(k)\}$ again, converging on any finite interval of Z^+ as $p \to \infty$. Thus we have a sequence $\{y_i(k)\}$ such that

$$x_{ip}(k)\to y_i(k),\quad u_{ip}(k)\to v_i(k)\ \text{for}\ k\in Z^+\ \text{as}\ p\to\infty.$$

This, combined with

$$\begin{cases} x_i(k+1+\delta_p) = x_i(k+\delta_p)\exp\left\{b_i(k+\delta_p) - \sum_{j=1}^{n} a_{ij}(k+\delta_p)(x_j(k+\delta_p))^{\theta_{ij}} - e_i(k+\delta_p)u_i(k+\delta_p)\right\}, \\ \Delta u_i(k+\delta_p) = -f_i(k+\delta_p)u_i(k+\delta_p) + \sum_{j=1}^{n} g_{ij}(k+\delta_p)x_j(k+\delta_p),\quad i=1,2,\cdots,n, \end{cases}$$

gives us

$$\begin{cases} y_i(k+1) = y_i(k)\exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(y_j(k))^{\theta_{ij}} - e_i(k)v_i(k)\right\}, \\ \Delta v_i(k) = -f_i(k)v_i(k) + \sum_{j=1}^{n} g_{ij}(k)y_j(k),\quad i=1,2,\cdots,n. \end{cases}$$

We can easily see that $(y_1(k), y_2(k), \cdots, y_n(k), v_1(k), v_2(k), \cdots, v_n(k))$ is a solution of system (1.1) and $m_i-\varepsilon \leq y_i(k) \leq M_i + \varepsilon$, $n_i - \varepsilon \leq v_i(k) \leq N_i + \varepsilon$ for $k \in Z^+$. Since ε is an arbitrary small positive number, it follows that $m_i \leq y_i(k) \leq M_i$, $n_i \leq v_i(k) \leq N_i$ and hence we complete the proof.

IV. Global Attractivity and Almost Periodic Solution

The main results of this paper concern the global attractivity of almost periodic solution of system (1.1) with conditions (1.2), (1.3) and (3.1).

Theorem 4.1 Assume that (1.2), (1.3), (3.1) and

$$\text{(H1)}\quad \rho_i = \max\{|1-\sum_{j=1}^{n}\theta_{ij}a_{ij}^l m_j^{\theta_{ij}}|,\ |1-\sum_{j=1}^{n}\theta_{ij}a_{ij}^u M_j^{\theta_{ij}}|\} + e_i^u < 1,$$
$$\varphi_i = 1 - f_i^l + \sum_{j=1}^{n} g_{ij}^u M_j < 1,\ i=1,2,\cdots,n,$$

hold. Then any positive solution $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ of system (1.1) is globally attractive.

Proof. Assume that $(p_1(k), p_2(k), \cdots, p_n(k), v_1(k), v_2(k), \cdots, v_n(k))$ is a solution of system (1.1) satisfying (1.2) and (1.3). Let

$$x_i(k) = p_i(k)\exp\{q_i(k)\},\quad u_i(k) = v_i(k) + w_i(k),\quad i=1,2,\cdots,n.$$

Since

$$\begin{aligned} q_i(k+1) &= \ln x_i(k+1) - \ln p_i(k+1) \\ &= \ln x_i(k) + b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(x_j(k))^{\theta_{ij}} - e_i(k)u_i(k) \\ &\quad - \ln p_i(k) - b_i(k) + \sum_{j=1}^{n} a_{ij}(k)(p_j(k))^{\theta_{ij}} + e_i(k)v_i(k) \\ &= q_i(k) - \sum_{j=1}^{n} a_{ij}(k)[(x_j(k))^{\theta_{ij}} - (p_j(k))^{\theta_{ij}}] - e_i(k)w_i(k) \\ &= q_i(k) - \sum_{j=1}^{n} a_{ij}(k)(p_j(k))^{\theta_{ij}}[(\exp\{q_j(k)\})^{\theta_{ij}} - 1] - e_i(k)w_i(k) \\ &= q_i(k)(1 - \sum_{j=1}^{n}\theta_{ij}a_{ij}(k)[p_j(k)\exp\{\lambda_j(k)q_j(k)\}]^{\theta_{ij}}) - e_i(k)w_i(k), \end{aligned}$$

where $\lambda_i(k)\in(0, 1)$.

Similarly, we get

$$\begin{aligned} w_i(k+1) &= u_i(k+1) - v_i(k+1) \\ &= (1-f_i(k))u_i(k) + \sum_{j=1}^{n} g_{ij}(k)x_j(k) - (1-f_i(k))v_i(k) - \sum_{j=1}^{n} g_{ij}(k)p_j(k) \\ &= (1-f_i(k))w_i(k) + \sum_{j=1}^{n} g_{ij}(k)p_j(k)(\exp\{q_j(k)\}-1) \\ &= (1-f_i(k))w_i(k) + \sum_{j=1}^{n} g_{ij}(k)p_j(k)q_j(k)\exp\{\xi_j(k)q_j(k)\},\quad i=1,2,\cdots,n, \end{aligned}$$

where $\xi_j(k)\in(0, 1)$.

To complete the proof, it suffices to show that

$$\lim_{k\to+\infty} q_i(k) = 0,\ \lim_{k\to+\infty} w_i(k) = 0,\quad i=1,2,\cdots,n. \tag{4.1}$$

In view of (H1), we can choose ε > 0 such that

$$\rho_i^\varepsilon = \max\{|1-\sum_{j=1}^{n}\theta_{ij}a_{ij}^l(m_j-\varepsilon)^{\theta_{ij}}|,\ |1-\sum_{j=1}^{n}\theta_{ij}a_{ij}^u(M_j+\varepsilon)^{\theta_{ij}}|\} + e_i^u < 1,$$
$$\varphi_i^\varepsilon = 1 - f_i^l + \sum_{j=1}^{n} g_{ij}^u(M_j+\varepsilon) < 1,\quad i=1,2,\cdots,n.$$

Let

$$\rho = \max\{\rho_i^\varepsilon, \varphi_i^\varepsilon\},$$

then ρ < 1. According to Theorem 3.2, there exists a positive integer $k_0 \in Z^+$ such that

$$m_i-\varepsilon\leq x_i(k)\leq M_i+\varepsilon,\quad m_i-\varepsilon\leq p_i(k)\leq M_i+\varepsilon,$$
$$i=1,2,\cdots,n$$

for $k \geq k_0$.

Notice that $\lambda_i(k)\in[0, 1]$ implies that $p_i(k)\exp\{\lambda_i(k)u_i(k)\}$ lies between $p_i(k)$ and $x_i(k)$, $\lambda_j(k) \in [0, 1]$ implies that $p_j(k)\exp\{\lambda_j(k)u_j(k)\}$ lies between $p_j(k)$ and $x_j(k)$. From (4.1), we get

$$\begin{aligned} |q_i(k+1)| \leq \max\{&|1-\sum_{j=1}^{n}\theta_{ij}a_{ij}^l(m_j-\varepsilon)^{\theta_{ii}}|, \\ &|1-\sum_{j=1}^{n}\theta_{ij}a_{ij}^u(M_j+\varepsilon)^{\theta_{ii}}|\}|q_i(k)| + e_i^u|w_i(k)|, \end{aligned}$$

$$|w_i(k+1)| \le (1-f_i^l)|w_i(k)| + \sum_{j=1}^{n} g_{ij}^u (M_j+\varepsilon)|q_j(k)|,$$

i = 1, 2, · · · , n, for $k \ge k_0$.

In view of (4.3), we get

$$\max\{\max_{1\le i\le n}|q_i(k+1)|, \max_{1\le i\le n}|w_i(k+1)|\} \le \rho\max\{\max_{1\le i\le n}|q_i(k)|, \max_{1\le i\le n}|w_i(k)|\}, \quad k \ge k_0.$$

This implies

$$\max\{\max_{1\le i\le n}|q_i(k)|, \max_{1\le i\le n}|w_i(k)|\} \le \rho^{k-k_0}\max\{\max_{1\le i\le n}|q_i(k_0)|, \max_{1\le i\le n}|w_i(k_0)|\}, \quad k \ge k_0.$$

Then (4.2) holds and we can obtain

$$\lim_{k\to+\infty}|x_i(k)-p_i(k)| = 0,$$

$$\lim_{k\to+\infty}|u_i(k)-v_i(k)| = 0, \quad i = 1,2,\cdots,n. \tag{4.2}$$

Therefore, positive solution $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ of system (1.1) is globally attractive.

Theorem 4.2 Assume that (1.2), (1.3), (3.1) and (H1) hold. Then system (1.1) admits a unique almost periodic solution which is globally attractive.

Proof. It follows from Proposition 3.1 that there exists a solution $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ of system (1.1) satisfying $m_i \le x_i(k) \le M_i$, $n_i \le u_i(k) \le N_i$, $k \in Z^+$.

Suppose that $(x_1(k), x_2(k), \cdots, x_n(k), u_1(k), u_2(k), \cdots, u_n(k))$ is any solution of system (1.1), then there exists an integer valued sequence $\{k'_p\}$, $k'_p \to +\infty$ as $p \to +\infty$, such that $(x_1(k + k'_p), x_2(k + k'_p), \cdots, x_n(k + k'_p), u_1(k + k'_p), u_2(k + k'_p), \cdots, u_n(k + k'_p))$ is a solution of the following system

$$\begin{cases} x_i(k+1) = x_i(k)\exp\left\{b_i(k+k'_p) - \sum_{j=1}^{n} a_{ij}(k+k'_p)(x_j(k))^{\theta_{ij}} - e_i(k+k'_p)u_i(k)\right\}, \\ \Delta u_i(k) = -f_i(k+k'_p)u_i(k) + \sum_{j=1}^{n} g_{ij}(k+k'_p)x_j(k), \quad i = 1,2,\cdots,n, \end{cases}$$

From above discussion and Theorem 3.2, we have that not only $(x_1(k+k'_p), x_2(k+k'_p), \cdots, x_n(k+k'_p), u_1(k+k'_p), u_2(k + k'_p), \cdots, u_n(k + k'_p))$ but also $(\Delta x_1(k + k'_p), \Delta x_2(k + k'_p), \cdots, \Delta x_n(k + k'_p), \Delta u_1(k + k'_p), \Delta u_2(k + k'_p), \cdots, \Delta u_n(k + k'_p))$ are uniformly bounded, thus $(x_1(k + k'_p), x_2(k + k'_p), \cdots, x_n(k + k'_p), u_1(k + k'_p), u_2(k + k'_p), \cdots, u_n(k + k'_p))$ are uniformly bounded and equi-continuous. By Ascoli's theorem[22], there exists a uniformly convergent subsequence $(x_1(k+k_p), x_2(k+k_p), \cdots, x_n(k+k_p), u_1(k+k_p), u_2(k+k_p), \cdots, u_n(k+k_p)) \subseteq (x_1(k + k'_p), x_2(k + k'_p), \cdots, x_n(k + k'_p), u_1(k + k'_p), u_2(k + k'_p), \cdots, u_n(k + k'_p))$ such that for any $\varepsilon > 0$, there exists a $k_0(\varepsilon) > 0$ with the property that if $m, n \ge k_0(\varepsilon)$ then

$$|x_i(k+k_m) - x_i(k+k_n)| < \varepsilon, \quad |u_i(k+k_m) - u_i(k+k_n)| < \varepsilon,$$

which shows from Lemma 2.2 that $(x_1(k+k_n), x_2(k+k_n), \cdots, x_n(k+k_n), u_1(k+k_n), u_2(k+k_n), \cdots, u_n(k+k_n))$ is asymptotically almost periodic sequence, then $(x_1(k + k_n), x_2(k + k_n), \cdots, x_n(k + k_n), u_1(k + k_n), u_2(k + k_n), \cdots, u_n(k+k_n))$ are the sum of an almost periodic sequence $(p_1(k+k_n), p_2(k+k_n), \cdots, p_n(k+k_n), v_1(k+k_n), v_2(k + k_n), \cdots, v_n(k + k_n))$ and a sequence $(q_1(k + k_n), q_2(k + k_n), \cdots, q_n(k + k_n), w_1(k + k_n), w_2(k + k_n), \cdots, w_n(k + k_n))$ defined on Z, such that

$$x_i(k+k_n) = p_i(k+k_n) + q_i(k+k_n),$$

$$u_i(k+k_n) = v_i(k+k_n) + w_i(k+k_n), \quad k \in Z,$$

where

$$\lim_{n\to+\infty} p_i(k+k_n) = p_i(k), \quad \lim_{n\to+\infty} v_i(k+k_n) = v_i(k),$$

$$\lim_{n\to+\infty} q_i(k+k_n) = 0, \quad \lim_{n\to+\infty} w_i(k+k_n) = 0,$$

$\{p_i(k)\}$ and $\{v_i(k)\}$ are almost periodic sequences, *i* = 1, 2, · · · , *n*. It means that

$$\lim_{n\to+\infty} x_i(k+k_n) = p_i(k), \quad \lim_{n\to+\infty} u_i(k+k_n) = v_i(k).$$

In the following we show that $\{(p_1(k), p_2(k), \cdots, p_n(k), v_1(k), v_2(k), \cdots, v_n(k))\}$ is an almost periodic solution of system (1.1).

From the properties of an almost periodic sequence, there exists an integer valued sequence $\{\delta_p\}$, $\delta_p \to +\infty$ as $p \to +\infty$, such that

$$b_i(k+\delta_p) \to b_i(k), \quad a_{ij}(k+\delta_p) \to a_{ij}(k),$$

$$e_i(k+\delta_p) \to e_i(k), \quad f_i(k+\delta_p) \to f_i(k),$$

$$g_{ij}(k+\delta_p) \to g_{ij}(k), \quad \text{as } p \to +\infty.$$

It is easy to know that

$$x_i(k+\delta_p) \to p_i(k), \; u_i(k+\delta_p) \to v_i(k)$$

then we have

$$\begin{aligned} p_i(k+1) &= \lim_{p\to\infty} x_i(k+1+\delta_p) \\ &= \lim_{p\to\infty} x_i(k+\delta_p)\exp\left\{b_i(k+\delta_p) - \sum_{j=1}^{n} a_{ij}(k+\delta_p)(x_j(k+\delta_p))^{\theta_{ij}} - e_i(k+\delta_p)u_i(k+\delta_p)\right\} \\ &= p_i(k)\exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(p_j(k))^{\theta_{ij}} - e_i(k)v_i(k)\right\}, \\ v_i(k+1) &= \lim_{p\to\infty} u_i(k+1+\delta_p) \\ &= \lim_{p\to\infty}\left\{[1-f_i(k+\delta_p)]u_i(k+\delta_p) + \sum_{j=1}^{n} g_{ij}(k+\delta_p)x_j(k+\delta_p)\right\} \\ &= [1-f_i(k)]v_i(k) + \sum_{j=1}^{n} g_{ij}(k)p_j(k). \end{aligned}$$

This prove that $p(k) = \{(p_1(k), p_2(k), \cdots, p_n(k), v_1(k), v_2(k), \cdots, v_n(k))\}$ satisfied system (1.1), and p(k) is a positive almost periodic solution of system (1.1).

Now, we show that there is only one positive almost periodic solution of system (1.1). For any two positive almost periodic solutions $(p_1(k), p_2(k), \cdots, p_n(k), v_1(k), v_2(k), \cdots, v_n(k))$ and $(z_1(k), z_2(k), \cdots, z_n(k), l_1(k), l_2(k), \cdots, l_n(k))$ of system (1.1), we claim that $p_i(k) = z_i(k)$, $v_i(k) = l_i(k)$ (i = 1, 2, · · · , n) for all $k \in Z^+$.

Otherwise there must be at least one positive integer $K^* \in Z^+$ such that $p_i(K^*) \ne z_i(K^*)$ or $v_j(K^*) \ne l_j(K^*)$ for a certain positive integer i or j, i.e., $\Omega_1 = |p_i(K^*) - z_i(K^*)| > 0$ or $\Omega_2 = |v_j(K^*) - l_j(K^*)| > 0$. So we can easily know that

$$\Omega_1 = |\lim_{p\to+\infty} p_i(K^*+\delta_p) - \lim_{p\to+\infty} z_i(K^*+\delta_p)| = \lim_{p\to+\infty}|p_i(K^*+\delta_p) - z_i(K^*+\delta_p)| = \lim_{k\to+\infty}|p_i(k) - z_i(k)| > 0,$$

or

$$\Omega_2 = |\lim_{p\to+\infty} v_j(K^*+\delta_p) - \lim_{p\to+\infty} l_j(K^*+\delta_p)| = \lim_{p\to+\infty}|v_j(K^*+\delta_p) - l_j(K^*+\delta_p)| = \lim_{k\to+\infty}|v_j(k) - l_j(k)| > 0,$$

which is a contradiction to (4.4). Thus $p_i(k) = z_i(k)$, $v_i(k) = l_i(k)$ (i = 1, 2, · · · , n) hold for $\forall k \in Z^+$. Therefore, system (1.1) admits a unique almost periodic solution which is globally attractive. This completes the proof of Theorem 4.2.

V. Example and Numerical Simulation

In this section, we give the following example to check the feasibility of our result.

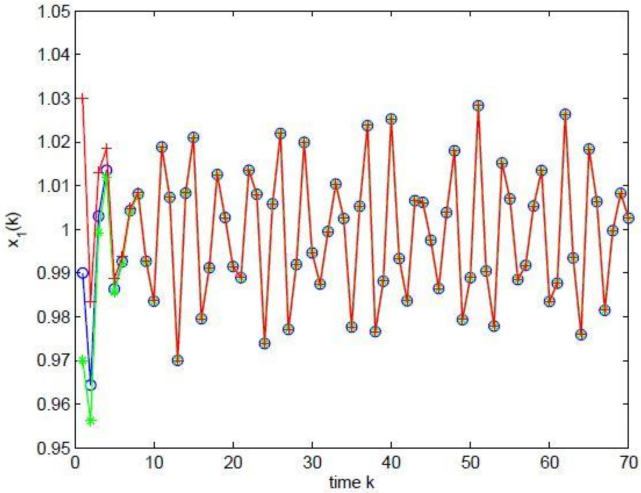
Example Consider the following almost periodic discrete Gilpin-Ayala competition system with feedback controls

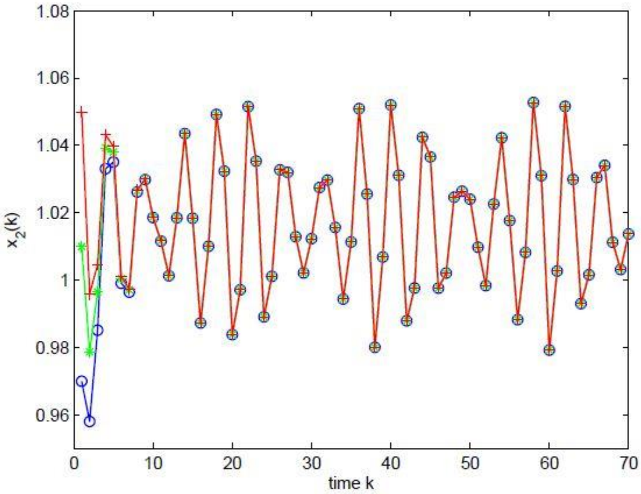
$$x_1(k+1)=x_1(k)\exp\Big\{1.1-0.022\sin(\sqrt{3}k)-(1.05+0.013\sin(\sqrt{5}k))x_1(k)$$
$$-(0.025-0.001\cos(\sqrt{2}k))x_2(k)-(0.02+0.0015\cos(\sqrt{3}k))x_3(k)-(0.025-0.002\cos(\sqrt{3}k))u_1(k)\Big\},$$
$$x_2(k+1)=x_2(k)\exp\Big\{1.15-0.025\sin(\sqrt{2}k)-(1.085+0.015\sin(\sqrt{3}k))x_2(k)$$
$$-(0.025+0.003\cos(\sqrt{5}k))x_1(k)-(0.025-0.002\cos(\sqrt{2}k))x_3(k)-(0.025+0.004\sin(\sqrt{2}k))u_2(k)\Big\},$$
$$x_3(k+1)=x_3(k)\exp\Big\{1.25-0.03\sin(\sqrt{5}k)-(1.1-0.024\sin(\sqrt{2}k))x_3(k) \tag{5.1}$$
$$-(0.03-0.002\cos(\sqrt{2}k))x_1(k)-(0.028+0.0015\cos(\sqrt{3}k))x_2(k)-(0.02+0.002\cos(\sqrt{3}k))u_3(k)\Big\},$$
$$\Delta u_1(k)=-(0.83-0.03\sin(\sqrt{2}k))u_1(k)+(0.015+0.005\sin(\sqrt{3}k))x_1(k)$$
$$+(0.013-0.004\sin(\sqrt{3}k))x_2(k)+(0.024-0.005\cos(\sqrt{5}k))x_3(k),$$
$$\Delta u_2(k)=-(0.824-0.04\sin(\sqrt{3}k))u_2(k)+(0.018-0.004\sin(\sqrt{5}k))x_1(k)$$
$$+(0.015-0.005\cos(\sqrt{2}k))x_2(k)+(0.014+0.004\sin(\sqrt{2}k))x_3(k),$$
$$\Delta u_3(k)=-(0.836-0.035\cos(\sqrt{5}k))u_3(k)+(0.017-0.006\cos(\sqrt{2}k))x_1(k)$$
$$+(0.013-0.005\sin(\sqrt{3}k))x_2(k)+(0.014+0.005\cos(\sqrt{2}k))x_3(k).$$

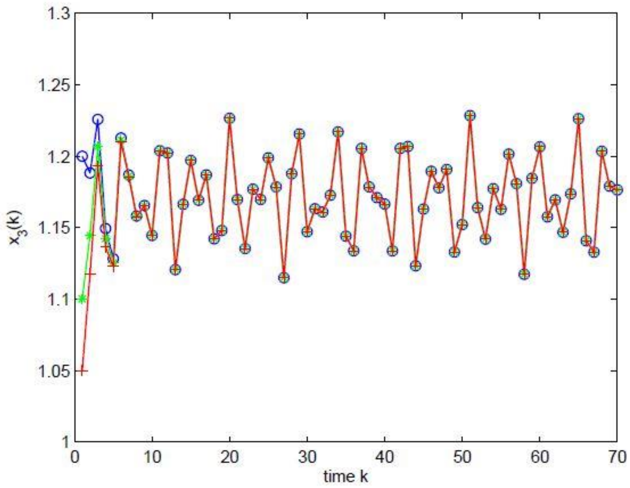
By simple computation, we derive

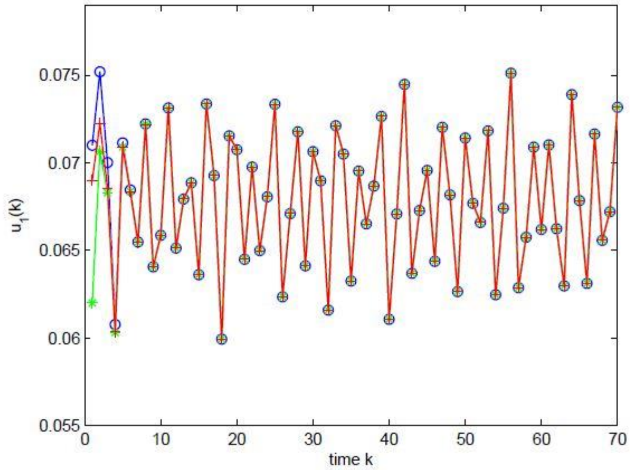
$$\rho_1\approx 0.193,\quad \rho_2\approx 0.341,\quad \rho_3\approx 0.298,$$
$$\varphi_1\approx 0.012,\quad \varphi_2\approx 0.075,\quad \varphi_2\approx 0.218.$$

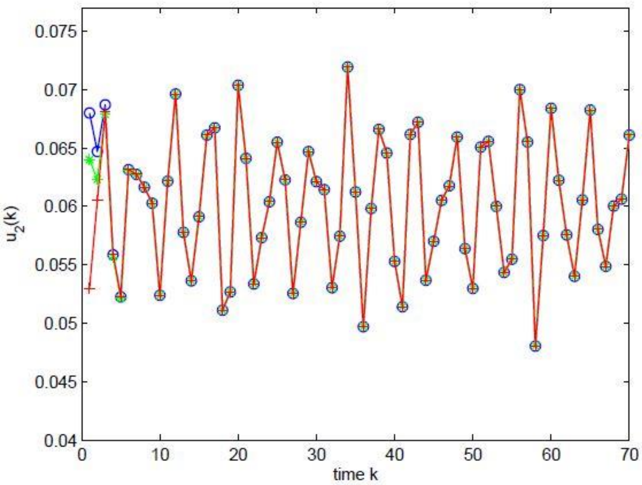
It is easy to see that the conditions of Theorem 4.2 are verified. Therefore, system (5.1) has a unique positive almost periodic solution which is globally attractive. Our numerical simulations support our results(see Figure1).











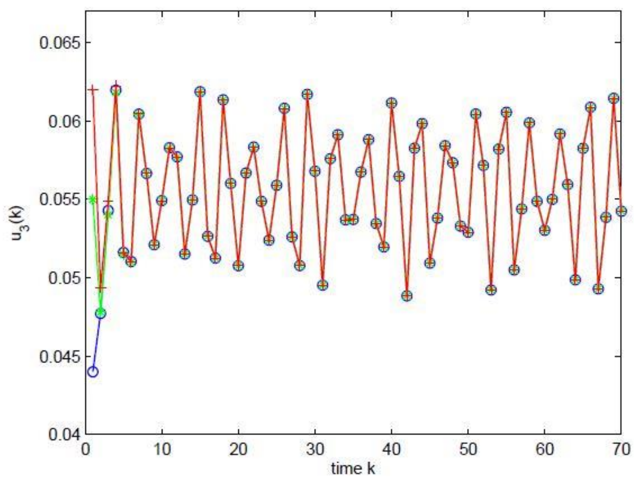


FIGURE1: Dynamic behavior of positive almost periodic solution ($x_1(k)$, $x_2(k)$, $x_3(k)$, $u_1(k)$, $u_2(k)$, $u_3(k)$) of system (5.1) with the three initial conditions(0.99,0.97,1.2,0.071, 0.068,0.044),(0.97,1.01,1.1,0.062,0.064,0.055) and (1.03, 1.05,1.05,0.069,0.053,0.062) for $k \in [1, 70]$, respectively.

VI. Concluding Remarks

In this paper, assuming that the coefficients in system (1.1) are bounded non-negative almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive. By comparative analysis, we find that when the coefficients in system (1.1) are almost periodic, the existence of a unique almost periodic solution of system (1.1) is determined by the global attractivity of system (1.1), which implies that there is no additional condition to add.

Furthermore, for the almost periodic discrete multispecies Gilpin-Ayala competition system with timedelays and feedback controls, we would like to mention here the question of how to study the almost periodicity of the system and whether the existence of a unique almost periodic solution is determined by the global attractivity of the system or not. It is, in fact, a very challenging problem, and we leave it for our future work.

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