Uniform asymptotical stability of almost periodic solution of a discrete multispecies Lotka-Volterra competition system

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Abstract— **In this paper, we study a discrete multispecies Lotka-Volterra competition system. Assume that the coefficients in the system are almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is uniformly asymptotically stable by constructing a suitable Liapunov function. One example together with numerical simulation indicates the feasibility of the main results.**

Index Terms—**Almost periodic solution, Discrete, Lotka-Volterra competition system, Permanence, Uniformly asymptotically stable**

I. INTRODUCTION

In paper [1], Chen and Wu had investigated the dynamic behavior of the following discrete n-species Gilpin-Ayala competition model

$$x_i(k+1) = x_i(k)\exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(x_j(k))^{\theta_{ij}}\right\}, \tag{1.1}$$

where $i = 1, 2, \cdots, n$; $x_i(k)$ is the density of competition species i at k-th generation. $a_{ij}(k)$ measures the intensity of intraspecific competition or interspecific action of competition species, respectively. $b_i(k)$ represents the intrinsic growth rate of the competition species x_i. θ_{ij} are positive constants. $b_i(k)$, $a_{ij}(k)$, $i, j = 1, 2, \cdots, n$ are all positive sequences bounded above and below by positive constants. Obviously, when $\theta_{ij} \equiv 1$, system (1.1) reduces to the traditional discrete multispecies Lotka-Volterra competition model

$$x_i(k+1) = x_i(k)\exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k)\right\}. \tag{1.2}$$

For general non-autonomous case, sufficient conditions which ensure the permanence and the global stability of system (1.1) and (1.2) are obtained; For periodic case, sufficient conditions which ensure the existence of an unique globally stable positive periodic solution of system (1.1) and (1.2) are obtained.

Notice that the investigation of almost periodic solutions for difference equations is one of most important topics in the qualitative theory of difference equations due to its applications in biology, ecology, neural network, and so forth(see [2–13] and the references cited therein). Wang and Liu [3] studied a discrete Lotka-Volterra competitive system

$$\begin{cases} x_1(n+1) = x_1(n)\exp\left[r_1(n) - a_1(n)x_1(n) - \dfrac{c_2(n)x_2(n)}{1+x_2(n)}\right], \\ x_2(n+1) = x_2(n)\exp\left[r_2(n) - a_2(n)x_2(n) - \dfrac{c_1(n)x_1(n)}{1+x_1(n)}\right], \quad n = 0, 1, 2, \cdots. \end{cases}$$

With the help of the methods of the Lyapunov function, some analysis techniques, and preliminary lemmas, they establish a criterion for the existence, uniqueness, and uniformly asymptotic stability of positive almost periodic solution of the system. However, few work has been done previously on an almost periodic version which is corresponding to system (1.2). Then, we will further investigate the global stability of almost periodic solution of system (1.2).

Denote as Z and Z^+ the set of integers and the set of nonnegative integers, respectively. For any bounded sequence $\{g(n)\}$ defined on Z, define

$$g^u = \sup_{n\in Z} g(n), g^l = \inf_{n\in Z} g(n).$$

Throughout this paper, we assume that:

(H1) $a_{ij}(k)$ and $b_i(k)$ are bounded positive almost periodic sequences such that

$$0 < a_{ij}^l \le a_{ij}(k) \le a_{ij}^u,\ 0 < b_i^l \le b_i(k) \le b_i^u,\ \ i, j = 1, 2, \cdots, n.$$

From the point of view of biology, in the sequel, we assume that $x(0) = (x_1(0), x_2(0), \cdots, x_n(0)) > 0$. Then it is easy to see that, for given $x(0) > 0$, the system (1.1) has a positive sequence solution $x(k) = (x_1(k), x_2(k), \cdots, x_n(k))(k \in Z^+)$ passing through $x(0)$.

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In Section 3, by applying the theory of difference inequality, we present the permanence results for system (1.2). In Section 4, we establish the sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution of system (1.2). The main results are illustrated by an example with a numerical simulation in the last section.

II. PRELIMINARIES

First, we give the definitions of the terminologies involved.

Definition 2.1([14]) A sequence x: Z→R is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : | x(n+\tau) - x(n) | < \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$| x(n+\tau) - x(n) | < \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of x(n).

Lemma 2.1([15]) If $\{x(n)\}$ is an almost periodic sequence, then $\{x(n)\}$ is bounded.

Lemma 2.2([16]) $\{x(n)\}$ is an almost periodic sequence if and only if, for any sequence $m_i \subset Z$, there exists a subsequence $\{m_{ik}\} \subset \{m_i\}$ such that the sequence $\{x(n + m_{ik})\}$

converges uniformly for all $n\in Z$ as $k\to\infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.3([15]) Suppose that $\{p_1(n)\}$ and $\{p_2(n)\}$ are almost periodic real sequences. Then $\{p_1(n)+p_2(n)\}$ and $\{p_1(n)p_2(n)\}$ are almost periodic;$1/p_1(n)$ is also almost periodic provided that $p_1(n)\neq 0$ for all $n\in Z$.

Moreover, if $\varepsilon > 0$ is an arbitrary real number, then there exists a relatively dense set that is ε−almost periodic common to $\{p_1(n)\}$ and $\{p_2(n)\}$.

Lemma 2.4([17]) Assume that sequence $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n+1)\le x(n)\exp\{a(n)-b(n)x(n)\}$$

for $n\in N$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n)\le \frac{1}{b^l}\exp\{a^u-1\}.$$

Lemma 2.5([17]) Assume that sequence $\{x(n)\}$ satisfies

$$x(n+1)\ge x(n)\exp\{a(n)-b(n)x(n)\},\quad n\ge N_0,$$

$$\limsup_{n\to+\infty} x(n)\le x^*$$

and $x(N_0)>0$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants and $N_0\in N$. Then

$$\liminf_{n\to+\infty} x(n)\ge \min\left\{\frac{a^l}{b^u}\exp\{a^l-b^u x^*\},\frac{a^l}{b^u}\right\}.$$

Consider the following almost periodic difference system:

$$x(n+1)=f(n,x(n)),\quad n\in Z^+, \tag{2.1}$$

where $f: Z^+\times S_B\to R^K$, $S_B=\{x\in R^k : \|x\| < B\}$, and $f(n, x)$ is almost periodic in n uniformly for $x\in S_B$ and is continuous in x. The product system of (2.1) is the following system:

$$x(n+1)=f(n,x(n)),\quad y(n+1)=f(n,y(n)), \tag{2.2}$$

and Zhang [18] obtained the following Theorem.

Theorem 2.6([18]) Suppose that there exists a Lyapunov function $V(n, x, y)$ defined for $n\in Z^+$, $\|x\| < B$, $\|y\| < B$ satisfying the following conditions:

(i) $a(\| x-y\|)\le V(n,x,y)\le b(\|x-y\|)$,
where $a,b\in K$ with $K=\{a\in C(R^+,R^+): a(0)=0$
and a is increasing};

(ii) $\| V(n,x_1,y_1)-V(n,x_2,y_2)\|\le L[\|x_1-x_2\| + \|y_1-y_2\|]$, where $L>0$ is a constant;

(iii) $\Delta V_{(2.2)}(n,x,y)\le -\alpha V(n,x,y)$, where $0<\alpha<1$ is a constant, and

$$\Delta V_{(2.2)}(n,x,y)\equiv V(n+1,f(n,x),f(n,y))-V(n,x,y).$$

Moreover, if there exists a solution $\varphi(n)$ of (2.1) such that $\|\varphi(n)\|\leqslant B^* < B$ for $n\in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of system (2.1) which is bounded by B^*. In particular, if $f(n, x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of system (2.1) of period ω.

III. Permanence

In this section, we establish a permanence result for system (1.2), which can be found by Lemma 2.4 and 2.5.

Proposition 3.1 Assume that (H1) holds. Then any positive solution $(x_1(k), x_2(k),\cdots, x_n(k))$ of system (1.2) satisfies

$$\limsup_{k\to+\infty} x_i(k)\le M_i, \tag{3.1}$$

where

$$M_i=\frac{1}{a_{ii}^l}\exp\{b_i^u-1\},\quad i=1,2,\cdots,n.$$

Proposition 3.2 Assume that (H1) and

$$\text{(H2)}\quad b_i^l-\sum_{j=1,j\neq i}^n a_{ij}^u M_j>0$$

hold for all $i = 1, 2,\cdots, n$, where M_i, $i = 1, 2,\cdots, n$ are defined by (3.1). Then for every solution $(x_1(k), x_2(k),\cdots, x_n(k))$ of system (1.1) satisfies

$$\liminf_{k\to+\infty} x_i(k)\ge m_i,$$

where

$$m_i=\min\left\{\frac{b_i^l-\sum_{j=1,j\neq i}^n a_{ij}^u M_j}{a_{ii}^u}\exp\left\{b_i^l-\sum_{j=1}^n a_{ij}^u M_j\right\},\ \frac{b_i^l-\sum_{j=1,j\neq i}^n a_{ij}^u M_j}{a_{ii}^u}\right\},\quad i=1,2,\cdots,n.$$

Theorem 3.3 Assume that (H1) and (H2) hold, then system (1.1) is permanent.

The next result tells us that there exist solutions of system (1.2) totally in the interval of Theorem 3.3. We denote by Ω the set of all solutions $(x_1(k), x_2(k),\cdots, x_n(k))$ of system (1.2) satisfying $m_i\leqslant x_i(k)\leqslant M_i (i=1, 2,\cdots, n)$ for all $k\in Z^+$.

Proposition 3.4 Assume that (H1) and (H2) hold. Then $\Omega\neq\Phi$.

Proof. By the almost periodicity of $\{a_{ij}(k)\}$ and $\{b_i(k)\}$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p\to+\infty$ as $p\to+\infty$ such that

$$a_{ij}(k+\delta_p)\to a_{ij}(k),\quad b_i(k+\delta_p)\to b_i(k),\quad \text{as}\quad p\to+\infty.$$

Let ε be an arbitrary small positive number. It follows from Theorem 3.3 that there exists a positive integer N_0 such that

$$m_i-\varepsilon\le x_i(k)\le M_i+\varepsilon,\quad k>N_0.$$

Write $x_{ip}(k) = x_i(k+\delta_p)$ for $k\geqslant N_0-\delta_p$ and $p = 1, 2,\cdots$. For any positive integer q, it is easy to see that there exists a sequence $\{x_{ip}(k) : p\geqslant q\}$ such that the sequence $x_p(k)$ has a subsequence, denoted by $\{x_{ip}(k)\}$ again, converging on any finite interval of Z as $p\to\infty$. Thus we have a sequence $\{y_i(k)\}$ such that

$$x_{ip}(k)\to y_i(k)\ \text{for}\ k\in Z^+\ \text{as}\ p\to+\infty.$$

This, combined with

$$x_i(k+1+\delta_p)=x_i(k+\delta_p)\exp\left\{b_i(k+\delta_p)-\sum_{j=1}^n a_{ij}(k+\delta_p)x_j(k+\delta_p)\right\}, i=1,2,\cdots,n$$

gives us

$$y_i(k+1)=y_i(k)\exp\left\{b_i(k)-\sum_{j=1}^n a_{ij}(k)y_j(k)\right\}, i=1,2,\cdots,n.$$

We can easily see that $(y_1(k), y_2(k),\cdots, y_n(k))$ is a solution of system (1.2) and $m_i-\varepsilon\leqslant y_i(k)\leqslant M_i+\varepsilon$ for $k\in Z^+$. Since ε is an arbitrarily small positive number, it follows that $m_i\leqslant y_i(k)\leqslant M_i$ and hence we complete the proof.

IV. ALMOST PERIODIC SOLUTION

The main results of this paper concern the existence of a unique uniformly asymptomatically stable almost periodic solution of system (1.2) by constructing a non-negative Lyapunov function.

Theorem 4.1 Assume that (H1), (H2) and

(H3) $0 < \beta = \min_{1 \le i \le n} \{\beta_i\} < 1$

hold, where

$$\beta_i = 2a_{ii}^l m_i - a_{ii}^{u2} M_i^2 - \sum_{j=1, j\neq i}^{n} \Big[M_i^2 a_{ji}^{u2} + (1 + a_{ii}^u M_i) a_{ij}^u M_j + (1 + a_{jj}^u M_j) a_{ji}^u M_i + M_i M_j \sum_{l=1, l\neq i,j}^{n} a_{li}^u a_{lj}^u \Big],$$

$i = 1, 2, \cdots, n$. Then there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.2) which is bounded by Ω for all $k \in Z^+$.

Proof. Let $p_i(k) = \ln x_i(k)$, $i = 1, 2, \cdots, n$. From system (1.2), we have

$$p_i(k+1) = p_i(k) + b_i(k) - \sum_{j=1}^{n} a_{ij}(k) e^{p_j(k)}, \quad i = 1, 2, \cdots, n. \tag{4.1}$$

From Proposition 3.4, we know that system (4.1) have bounded solution $(p_1(k), p_2(k), \cdots, p_n(k))$ satisfying

$$\ln m_i \le p_i(k) \le \ln M_i, \quad i = 1, 2, \cdots, n, \quad k \in Z^+.$$

Hence, $|p_i(k)| \leqslant A_i$, where $A_i = \max\{|\ln m_i|, |\ln M_i|\}$, $i = 1, 2, \cdots, n$.

For $X \in R^n$, we define the norm $\|X\| = \sum_{i=1}^{n} x_i$.

Consider the product system of system (4.1)

$$\begin{cases} p_i(k+1) = p_i(k) + b_i(k) - \sum_{j=1}^{n} a_{ij}(k) e^{p_j(k)}, \\ q_i(k+1) = q_i(k) + b_i(k) - \sum_{j=1}^{n} a_{ij}(k) e^{q_j(k)}, \quad i = 1, 2, \cdots, n. \end{cases} \tag{4.2}$$

We assume that $Q = (p_1(k), p_2(k), \cdots, p_n(k))$, $W = (q_1(k), q_2(k), \cdots, q_n(k))$ are any two solutions of system (4.1) defined on $Z^+ \times S^*$; then, $\|Q\| \leqslant B$, $\|W\| \leqslant B$, where $B = \sum_{i=1}^{n} \{A_i + B_i\}$, $S^* = \{(p_1(k), p_2(k), \cdots, p_n(k)) | \ln m_i \leqslant p_i(n) \leqslant \ln M_i, i = 1, 2, \cdots, n, k \in Z^+\}$.

Let us construct a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(k, Q, W) = \sum_{i=1}^{n} (p_i(k) - q_i(k))^2.$$

It is obvious that the norm $\|Q-W\| = \sum_{i=1}^{n} |p_i(k)-q_i(k)|$ is equivalent to $\|Q-W\|_* = [\sum_{i=1}^{n} (p_i(k)-q_i(k))^2]^{1/2}$; that is, there are two constants $c_1 > 0$, $c_2 > 0$, such that

$$c_1 \|Q - W\| \le \|Q - W\|_* \le c_2 \|Q - W\|,$$

then

$$(c_1 \|Q - W\|)^2 \le V(k, Q, W) \le (c_2 \|Q - W\|)^2.$$

Let

$$\psi, \varphi \in C(R^+, R^+), \psi(x) = c_1^2 x^2, \varphi(x) = c_2^2 x^2;$$

then, condition (i) of Theorem 2.6 is satisfied.

Moreover, for any

$$(k, Q, W), (k, \overline{Q}, \overline{W}) \in Z^+ \times S^* \times S^*,$$

we have

$$\begin{aligned} &|V(k, Q, W) - V(k, \overline{Q}, \overline{W})| \\ &= \Big| \sum_{i=1}^{n} (p_i(k) - q_i(k))^2 - \sum_{i=1}^{n} (\overline{p_i}(k) - \overline{q_i}(k))^2 \Big| \\ &\le \sum_{i=1}^{n} |(p_i(k) - q_i(k))^2 - (\overline{p_i}(k) - \overline{q_i}(k))^2| \\ &= \sum_{i=1}^{n} |(p_i(k) - q_i(k)) + (\overline{p_i}(k) - \overline{q_i}(k))| |(p_i(k) - q_i(k)) - (\overline{p_i}(k) - \overline{q_i}(k))| \\ &\le \sum_{i=1}^{n} (|p_i(k)| + |q_i(k)| + |\overline{p_i}(k)| + |\overline{q_i}(k)|)(|p_i(k) - \overline{p_i}(k)| + |q_i(k) - \overline{q_i}(k)|) \\ &\le L \Big[\sum_{i=1}^{n} |p_i(k) - \overline{p_i}(k)| + \sum_{i=1}^{n} |q_i(k) - \overline{q_i}(k)| \Big] \\ &= L(\|Q - \overline{Q}\| + \|W - \overline{W}\|), \end{aligned}$$

where

$$\overline{Q} = (\overline{p_1}(k), \overline{p_2}(k), \cdots, \overline{p_n}(k)), \ \overline{W} = (\overline{q_1}(k), \overline{q_2}(k), \cdots, \overline{q_n}(k)),$$

and

$$L = 4 \max\{ \max_{1 \le i \le n} \{A_i\}, \max_{1 \le i \le n} \{B_i\} \}.$$

Thus, condition (ii) of Theorem 2.6 is satisfied.

Finally, calculating the $\Delta V(k)$ of $V(k)$ along the solutions of system (4.2), we have

$$\begin{aligned} &\Delta V_{(4.2)}(k) = V(k+1) - V(k) \\ &= \sum_{i=1}^{n} (p_i(k+1) - q_i(k+1))^2 - \sum_{i=1}^{n} (p_i(k) - q_i(k))^2 \\ &= \sum_{i=1}^{n} [(p_i(k+1) - q_i(k+1))^2 - (p_i(k) - q_i(k))^2] \\ &= \sum_{i=1}^{n} \Big\{ \Big[(p_i(k) - q_i(k)) - a_{ii}(k)(e^{p_i(k)} - e^{q_i(k)}) + \sum_{j=1, j\neq i}^{n} a_{ij}(k)(e^{p_j(k)} - e^{q_j(k)}) \Big]^2 \\ &\quad - (p_i(k) - q_i(k))^2 \Big\} \\ &= \sum_{i=1}^{n} \Big\{ -2a_{ii}(k)(p_i(k) - q_i(k))(e^{p_i(k)} - e^{q_i(k)}) + a_{ii}^2(k)(e^{p_i(k)} - e^{q_i(k)})^2 \\ &\quad - 2a_{ii}(k)(e^{p_i(k)} - e^{q_i(k)}) \sum_{j=1, j\neq i}^{n} a_{ij}(k)(e^{p_j(k)} - e^{q_j(k)}) \\ &\quad + \Big(\sum_{j=1, j\neq i}^{n} a_{ij}(k)(e^{p_j(k)} - e^{q_j(k)}) \Big)^2 \\ &\quad + 2(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} a_{ij}(k)(e^{p_j(k)} - e^{q_j(k)}) \Big\}. \end{aligned}$$

By the mean value theorem, it derives that

$$e^{p_i(k)} - e^{q_i(k)} = \xi_i(k)(p_i(k) - q_i(k)), \ i = 1, 2, \cdots, n,$$

where $\xi_i(k)$ lies between $e^{p_i(k)}$ and $e^{q_i(k)}$. Then, we have

$$\begin{aligned} &\Delta V_{(4.2)}(k) = \sum_{i=1}^{n} \Big\{ -2a_{ii}(k)\xi_i(k)(p_i(k) - q_i(k))^2 + a_{ii}^2(k)\xi_i^2(k)(p_i(k) - q_i(k))^2 \\ &\quad - 2a_{ii}(k)\xi_i(k)(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} a_{ij}(k)\xi_j(k)(p_j(k) - q_j(k)) \\ &\quad + \Big(\sum_{j=1, j\neq i}^{n} a_{ij}(k)\xi_j(k)(p_j(k) - q_j(k)) \Big)^2 \\ &\quad + 2(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} a_{ij}(k)\xi_j(k)(p_j(k) - q_j(k)) \Big\} \\ &= \sum_{i=1}^{n} \Big\{ \Big(-2a_{ii}(k)\xi_i(k) + a_{ii}^2(k)\xi_i^2(k) + \sum_{j=1, j\neq i}^{n} a_{ji}^2(k)\xi_i^2(k) \Big)(p_i(k) - q_i(k))^2 \\ &\quad + 2 \sum_{j=1, j\neq i}^{n} \Big([1 - a_{ii}(k)\xi_i(k)] a_{ij}(k)\xi_j(k) + \frac{1}{2} \sum_{l=1, l\neq i,j}^{n} a_{li}(k)\xi_i(k) a_{lj}(k)\xi_j(k) \Big) \times \\ &\quad (p_i(k) - q_i(k))(p_j(k) - q_j(k)) \Big\} \end{aligned}$$

$$\leq \sum_{i=1}^{n}\Big\{\Big(-2a_{ii}(k)\xi_i(k)+a_{ii}^2(k)\xi_i^2(k)+\sum_{j=1,j\neq i}^{n}a_{ji}^2(k)\xi_i^2(k)\Big)(p_i(k)-q_i(k))^2$$
$$+2\Big|\sum_{j=1,j\neq i}^{n}\Big([1-a_{ii}(k)\xi_i(k)]a_{ij}(k)\xi_j(k)+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}a_{li}(k)\xi_i(k)a_{lj}(k)\xi_j(k)\Big)\times$$
$$(p_i(k)-q_i(k))(p_j(k)-q_j(k))\Big|\Big\}.$$

Then, we have

$$\Delta V_{(4.2)}(k)\leq\sum_{i=1}^{n}[V_{i1}(k)+V_{i2}(k)],$$

where

$$V_{i1}(k)=\Big(-2a_{ii}(k)\xi_i(k)+a_{ii}^2(k)\xi_i^2(k)+\sum_{j=1,j\neq i}^{n}a_{ji}^2(k)\xi_i^2(k)\Big)(p_i(k)-q_i(k))^2$$
$$\leq\Big(-2a_{ii}^l m_i+a_{ii}^{u2}M_i^2+M_i^2\sum_{j=1,j\neq i}^{n}a_{ji}^{u2}\Big)(p_i(k)-q_i(k))^2,$$

$$V_{i2}(k)=2\Big|\sum_{j=1,j\neq i}^{n}\Big([1-a_{ii}(k)\xi_i(k)]a_{ij}(k)\xi_j(k)+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}a_{li}(k)\xi_i(k)a_{lj}(k)\xi_j(k)\Big)\times$$
$$(p_i(k)-q_i(k))(p_j(k)-q_j(k))\Big|$$
$$\leq\sum_{j=1,j\neq i}^{n}\Big((1+a_{ii}^uM_i)a_{ij}^uM_j+\frac{1}{2}M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big)[(p_i(k)-q_i(k))^2+(p_j(k)-q_j(k))^2]$$

Hence, we have

$$\Delta V_{(4.2)}(k)\leq\sum_{i=1}^{n}\Big\{\Big(-2a_{ii}^lm_i+a_{ii}^{u2}M_i^2+\sum_{j=1,j\neq i}^{n}\Big[M_i^2a_{ji}^{u2}+(1+a_{ii}^uM_i)a_{ij}^uM_j+\frac{1}{2}M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big]\Big)\times$$
$$(p_i(k)-q_i(k))^2+\sum_{j=1,j\neq i}^{n}\Big((1+a_{ii}^uM_i)a_{ij}^uM_j+\frac{1}{2}M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big)(p_j(k)-q_j(k))^2\Big\}$$
$$=\sum_{i=1}^{n}\Big\{\Big(-2a_{ii}^lm_i+a_{ii}^{u2}M_i^2+\sum_{j=1,j\neq i}^{n}\Big[M_i^2a_{ji}^{u2}+(1+a_{ii}^uM_i)a_{ij}^uM_j+\frac{1}{2}M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big]\Big)(p_i(k)-q_i(k))^2$$
$$+\sum_{j=1,j\neq i}^{n}\Big((1+a_{jj}^uM_j)a_{ji}^uM_i+\frac{1}{2}M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big)(p_i(k)-q_i(k))^2\Big\}$$
$$=\sum_{i=1}^{n}\Big\{\Big(-2a_{ii}^lm_i+a_{ii}^{u2}M_i^2+\sum_{j=1,j\neq i}^{n}\Big[M_i^2a_{ji}^{u2}+(1+a_{ii}^uM_i)a_{ij}^uM_j+(1+a_{jj}^uM_j)a_{ji}^uM_i+M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big]\Big)\times$$
$$(p_i(k)-q_i(k))^2\Big\}$$
$$=-\sum_{i=1}^{n}\Big\{\Big(2a_{ii}^lm_i-a_{ii}^{u2}M_i^2-\sum_{j=1,j\neq i}^{n}\Big[M_i^2a_{ji}^{u2}+(1+a_{ii}^uM_i)a_{ij}^uM_j+(1+a_{jj}^uM_j)a_{ji}^uM_i+M_iM_j\sum_{l=1,l\neq i,j}^{n}a_{li}^ua_{lj}^u\Big]\Big)\times$$
$$(p_i(k)-q_i(k))^2\Big\}$$
$$\leq-\sum_{i=1}^{n}\beta_i(p_i(k)-q_i(k))^2$$
$$\leq-\beta\sum_{i=1}^{n}(p_i(k)-q_i(k))^2$$
$$=-\beta V(k,Q,W),$$

where $\beta=\min_{1\leq i\leq n}\{\beta_i\}$. That is, there exists a positive constant $0<\beta<1$ such that

$$\Delta V_{(4.2)}(k,Q,W)\leq-\beta V(k,Q,W).$$

From $0<\beta<1$, the condition (iii) of Theorem 2.6 is satisfied. So, according to Theorem 2.6, there exists a unique uniformly asymptotically stable almost periodic solution $(p_1(k), p_2(k), \cdots, p_n(k))$ of system (4.1) which is bounded by S^* for all $k \in Z^+$. It means that there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.2) which is bounded by Ω for all $k \in Z^+$. This completed the proof. 2

Remark 4.2 If n = 2, the conditions of Theorem 4.1 can be simplified. Therefore, we have the following results.

Corollary 4.3 Let n = 2, assume that (H1), (H2) and

$$0<\beta=\min\{\beta_{12},\beta_{21}\}<1$$

hold, where

$$\beta_{ij}=2a_{ii}^lm_i-a_{ii}^{u2}M_i^2-M_i^2a_{ji}^{u2}-(1+2a_{ii}^uM_i)a_{ij}^uM_j$$
$$-(1+2a_{jj}^uM_j)a_{ji}^uM_i,$$

i, j = 1, 2, $j\neq i$. Then there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k))$ of system (1.2) which is bounded by Ω for all $k \in Z^+$.

V. NUMERICAL SIMULATION

In this section, we give the following examples to check the feasibility of our results.

Example 5.1 Consider the discrete multispecies Lotka-Volterra competition system:

$$\begin{cases} x_1(k+1)=x_1(k)\exp\Big\{1.1-0.01\sin(\sqrt{3}k)-(1.15-0.01\sin(\sqrt{2}k))x_1(k) \\ \qquad -(0.055+0.002\cos(\sqrt{5}k))x_2(k)-(0.03+0.002\cos(\sqrt{5}k))x_3(k)\Big\}, \\ x_2(k+1)=x_2(k)\exp\Big\{1.1-0.025\sin(\sqrt{3}k)-(0.02-0.003\cos(\sqrt{5}k))x_1(k) \\ \qquad -(1.08+0.025\sin(\sqrt{2}k))x_2(k)-(0.025+0.002\cos(\sqrt{5}k))x_3(k)\Big\}, \\ x_3(k+1)=x_3(k)\exp\Big\{1.15-0.02\sin(\sqrt{2}k)-(0.04+0.0025\cos(\sqrt{3}k))x_1(k) \\ \qquad -(0.029+0.0012\sin(\sqrt{2}k))x_2(k)-(1.13+0.02\sin(\sqrt{5}n))x_3(k)\Big\}. \end{cases} \tag{5.1}$$

A computation shows that

$$m_1\approx0.7743,\ M_1\approx0.9807,\ m_2\approx0.9109,$$
$$M_2\approx1.1572,\ m_3\approx0.8123,\ M_3\approx1.0579,$$

and moreover, we have

$$\beta_1\approx0.0652,\ \beta_2\approx0.0297,\ \beta_3\approx0.0451,$$

that $0<\min\{\beta_1,\beta_2,\beta_3\}<1$. It is easy to see that the condition (H2) and (H3) are satisfied. Hence, there exists a unique uniformly asymptotically stable almost periodic solution of system (5.1). Our numerical simulations support our results(see Figs.1,2 and 3).

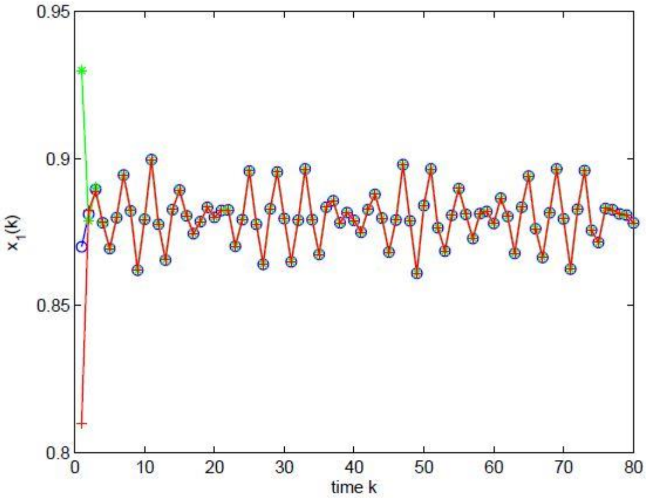

FIGURE1: Dynamic behavior of the first component $x_1(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (5.1) with the initial conditions (0.87,1.02,1.03), (0.93,1.13,0.86) and (0.81,0.97,0.97) for $k \in [1, 80]$, respectively.

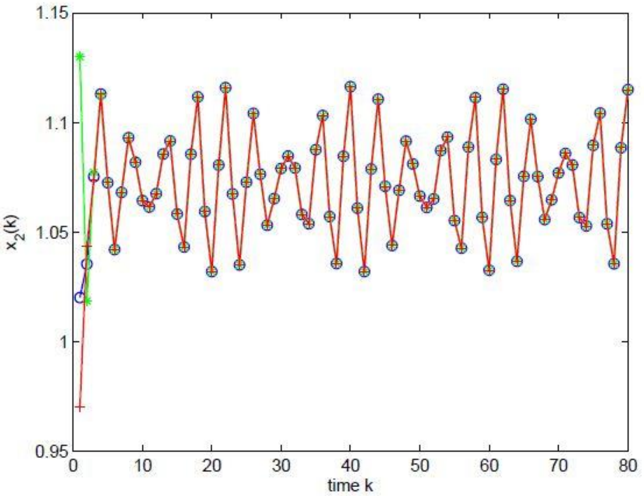

FIGURE2: Dynamic behavior of the second component $x_2(k)$

of the solution ($x_1(k)$, $x_2(k)$, $x_3(k)$) to system (5.1) with the initial conditions (0.87,1.02,1.03), (0.93,1.13,0.86) and (0.81,0.97,0.97) for k ∈ [1, 80], respectively.

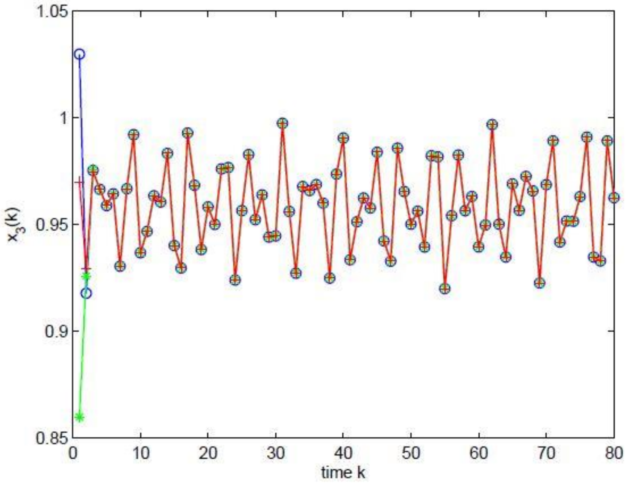

FIGURE3: Dynamic behavior of the third component $x_3(k)$ of the solution ($x_1(k)$, $x_2(k)$, $x_3(k)$) to system (5.1) with the initial conditions (0.87,1.02,1.03), (0.93,1.13,0.86) and (0.81,0.97,0.97) for k ∈ [1, 80], respectively.

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